THE REPUBLIC OF UGANDA

Ministry of Education and Sports

HOME-STUDY LEARNING

SENIOR 6

MATHEMATICS

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NCDC
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This material has been developed as a home-study intervention for schools during the lockdown caused by the COVID-19 pandemic to support continuity of learning.

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FOREWORD

Following the outbreak of the COVID-19 pandemic, government of Uganda closed all schools and other educational institutions to minimize the spread of the coronavirus. This has affected more than 36,314 primary schools, 3129 secondary schools, 430,778 teachers and 12,777,390 learners.

The COVID-19 outbreak and subsequent closure of all has had drastically impacted on learning especially curriculum coverage, loss of interest in education and learner readiness in case schools open. This could result in massive rates of learner dropouts due to unwanted pregnancies and lack of school fees among others.

To mitigate the impact of the pandemic on the education system in Uganda, the Ministry of Education and Sports (MoES) constituted a Sector Response Taskforce (SRT) to strengthen the sector's preparedness and response measures. The SRT and National Curriculum Development Centre developed print home-study materials, radio and television scripts for some selected subjects for all learners from Pre-Primary to Advanced Level. The materials will enhance continued learning and learning for progression during this period of the lockdown, and will still be relevant when schools resume.

The materials focused on critical competences in all subjects in the curricula to enable the learners to achieve without the teachers' guidance. Therefore effort should be made for all learners to access and use these materials during the lockdown. Similarly, teachers are advised to get these materials in order to plan appropriately for further learning when schools resume, while parents/guardians need to ensure that their children access copies of these materials and use them appropriately.
I recognise the effort of National Curriculum Development Centre in responding to this emergency through appropriate guidance and the timely development of these home study materials. I recommend them for use by all learners during the lockdown.

Alex Kakooza
Permanent Secretary
Ministry of Education and Sports

ACKNOWLEDGEMENTS

National Curriculum Development Centre (NCDC) would like to express its appreciation to all those who worked tirelessly towards the production of home–study materials for Pre-Primary, Primary and Secondary Levels of Education during the COVID-19 lockdown in Uganda.

The Centre appreciates the contribution from all those who guided the development of these materials to make sure they are of quality; Development partners - SESIL, Save the Children and UNICEF; all the Panel members of the various subjects; sister institutions - UNEB and DES for their valuable contributions.

NCDC takes the responsibility for any shortcomings that might be identified in this publication and welcomes suggestions for improvement. The comments and suggestions may be communicated to NCDC through P.O. Box 7002 Kampala or email admin@ncdc.go.ug or by visiting our website at http://ncdc.go.ug/node/13.

Grace K. Baguma
Director,
National Curriculum Development Centre

ABOUT THIS BOOKLET

Dear learner, you are welcome to this home-study package. This content focuses on critical competences in the syllabus.

The content is organised into lesson units. Each unit has lesson activities, summary notes and assessment activities. Some lessons have projects that you need to carry out at home during this period. You are free to use other reference materials to get more information for specific topics.

Seek guidance from people at home who are knowledgeable to clarify in case of a challenge. The knowledge you can acquire from this content can be supplemented with other learning options that may be offered on radio, television, newspaper learning programmes. More learning materials can also be accessed by visiting our website at www.ncdc.go.ug or ncdc-go-ug.digital/. You can access the website using an internet enabled computer or mobile phone.

We encourage you to present your work to your class teacher when schools resume so that your teacher is able to know what you learned during the time you have been away from school. This will form part of your assessment. Your teacher will also assess the assignments you will have done and do corrections where you might not have done it right.

The content has been developed with full awareness of the home learning environment without direct supervision of the teacher. The methods, examples and activities used in the materials have been carefully selected to facilitate continuity of learning.

You are therefore in charge of your own learning. You need to give yourself favourable time for learning. This material can as well be used beyond the home-study situation. Keep it for reference anytime.

Develop your learning timetable to ca ter for continuity of learning and other responsibilities given to you at home.

Enjoy learning

Class: S6 Pure Mathematics: Term 1

Topic 12: Trigonometry (Calculus)

Learning Outcomes

The learner should be able to:

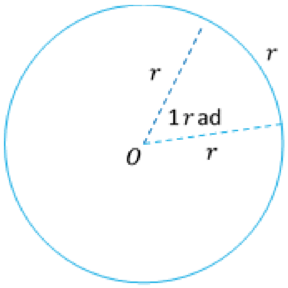
i) convert degrees to radians and vice versa.

ii) find the value of the length of an arc and the area of a sector.

Sub-topic 1: Radians

Lesson: Radians

If an arc AB of a circle is drawn so that it is equal in length to the radius of the circle, then the angle is called one radian (1 rad)



The number of radians in one revolution is therefore given by the ratio of

$circumference = \frac{2\pi r}{r} = 2\pi\ radians;\ 2\pi = 360\ degrees$

- $\pi = 180\ degrees;\ \frac{\pi}{180} radians = 1\ degree$

Example 1

(a)Express the following in radians (Exercise 18b Qn. 1 & 2)

i) $30^0 = 30 \times \frac{\pi}{180} = \frac{\pi}{6}$

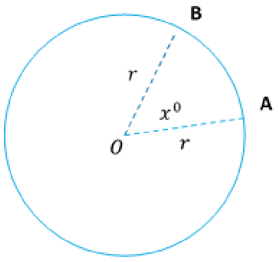
ii) $45^0 = 45 \times \frac{\pi}{180} = \frac{\pi}{4}$

iii) $120^0 30' = (120 + 30/60) \times \frac{\pi}{180} = 120.5 \times \frac{\pi}{180}$

iv) $12' = \frac{12}{60} \times \frac{\pi}{180} = \frac{0.1\pi}{90} = \frac{\pi}{900}$

(b)Express the following in degrees; $\frac{5\pi}{6} = \frac{5\pi}{6} \times \frac{180}{\pi} = 150^0$

Example 2.



An arc AB of a circle, centre O subtends an angle x^0 at O. find the expression in terms of X and the radius for the

a) For the length of the arc AB

b) The area of the sector OAB

a) The length of an arc of a given circle is proportional to the angle it subtends at the centre. But an angle of is subtended by an arc of length$2\pi r$, therefore an angle of x^o is subtended by an arc of length.

The length of Arc= $= \frac{x^0}{360^0} \times 2\pi r = \frac{\pi}{180} \times r$

$$\therefore The\ length\ of\ the\ sector\ OAB\ is\ = \left(\frac{\pi}{180}\right) xr$$

b) $Area\ of\ sector\ OAB = \frac{x^0}{360^0} \times \pi \times r^2 = \frac{1}{2}\left(\frac{\pi}{180}\right) xr^2$

Thus in both the length of an arc and the area of a sector, there appears a factor of$\frac{\pi}{180}$, which is due to the unit of measurement of the angle OAB. This suggests a new unit for measuring angles, which is called **a radian**, such that an angle in radius $= \frac{\pi}{180}$ (angle in degrees)

If we let θ radians equals $degrees$, then$1^0 = \frac{\pi}{180}\ rad;\ x^0 = \frac{\pi}{180} x\ radians = \theta\ radians$

$\therefore The\ length\ of\ the\ sector\ OAB\ is\ = \left(\frac{\pi}{180}\right) xr = \theta r = r\theta;$

$\therefore Area\ of\ sector\ OAB = \frac{1}{2}\left(\frac{\pi}{180}\right) xr^2 = \frac{1}{2}\theta r^2 = \frac{1}{2} r^2\theta$

Exercise 1

1.(a) What is the length of an arc which subtends an angle 0.8 rad at rad at the centre of circle of radius 10 cm?

(b) What is the area of a sector containing an angle of 1.5 rad in a circle radius 2cm?

2. Two equal circles of radius 5cm are situated with their centres 6cm apart. Calculate what area lies within both circles. Area of shaded part = 2 (Area of sector OAB – Area of triangle OAB) = 22.36 cm^2

3. A circle of radius r is drawn with its centre on the circumference of another circle of radius r. show that the area common to both circles is $2r^2\left(\frac{\pi}{3} - \frac{\sqrt{3}}{4}\right)$

Sub-topic 2: Derivatives of Trigonometrical Functions

Learning Outcomes

The learner should be able to:

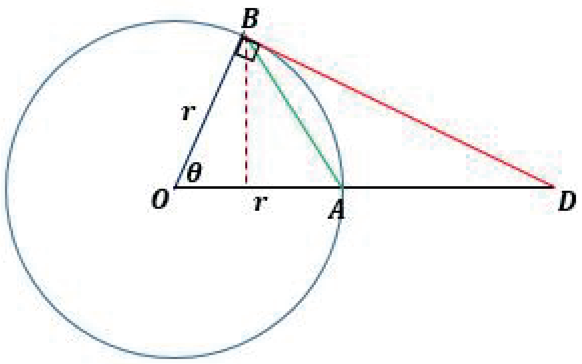
i) find a relationship between $sin\theta$, $cos\theta$, $\tan\theta$ and $\theta(rads)$ for θ is a small angle.
ii) differentiate the trigonometrical ratios from first principles.
iii) differentiate the inverse trigonometrical functions.

Lesson: Derivatives of Trigonometrical Functions

Small angles

For very small acute angles $\tan\theta \approx \sin\theta \approx \theta$, i.e.

Angles in degrees		5^0	1^0
θ *radians*		0.087266462	0.017453292
$\tan\theta$		0.087488663	0.017455064
$\sin\theta$	0.087155742	0.017452406	



Consider this geometrically; Chord AB subtends an angle θ at the centre of a circle of radius r and the tangent at B meet OA at D. consider the three areas of triangle AOB sector AOB and triangle DOB

.Area of Δ $triangle\ AOB = \frac{1}{2}r^2 sin\,\theta$

$Area\ of\ Sector\ AOB = \frac{1}{2}r^2\theta;$

Area of Δ $triangle\ DOB = \frac{1}{2}r^2 tan\,\theta$

It can be seen that $Area\ of\ AOB < sector\ of\ OAB < area\ of\ \Delta\, DBO$

$\frac{1}{2}r^2\sin\theta < \frac{1}{2}r^2\theta < \frac{1}{2}r^2\tan\theta$

Divide through by $\frac{1}{2}r^2\sin\theta$

$\frac{\text{Sin}\,\theta}{\sin\theta} < \frac{\theta}{\sin\theta} < \frac{\sin\theta}{\cos\theta\sin\theta}$;

$\rightarrow 1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}\ as\ \theta \rightarrow 0, \cos\theta \rightarrow 1,\ \ \frac{1}{\cos\theta} \rightarrow 1$

$1 < \frac{\theta}{\sin\theta} < 1\ ;\ \frac{\theta}{\sin\theta} \approx 1\ as\ \theta \rightarrow 0$

$;i.e\ \sin\theta \approx \theta\ as\ \theta \rightarrow 0$

Find the approximate value of Cos θ when θ is very small.

$\cos\theta = 1 - 2\sin^2\left(\frac{\theta}{2}\right)$; $\sin\frac{\theta}{2} \approx \frac{\theta}{2}$; $\cos\theta \approx 1 - \frac{2\theta^2}{4} = 1 - \frac{\theta^2}{2}$

Derivative of sin x and cos x

1) $y = \sin x$

$y + \delta y = \sin(x + \delta x)$; $\delta y = \sin(x+\delta x) - y = \sin(x+\delta x) - \sin x$

$\delta y = 2\cos\left(\frac{x+\delta x+x}{2}\right)\sin\left(\frac{x+\delta x-x}{2}\right) = 2\cos\left(x + \frac{\delta x}{2}\right)\sin\left(\frac{\delta x}{2}\right)$

Dividing by δx; $\frac{\delta y}{\delta x} = \frac{\cos\left(x+\frac{\delta x}{2}\right)\sin\left(\frac{\delta x}{2}\right)}{{}^{1}/{}_{2}\delta x}$

The limit as $\delta x \to 0$, $\frac{\delta y}{\delta x} \to \frac{dy}{dx}$, $\sin\left(\frac{\delta x}{2}\right) \approx \frac{\delta x}{2}$; *Therefore* $\cos\left(x + \frac{\delta x}{2}\right) \approx \cos x$; $\frac{dy}{dx} = \cos x$

$\therefore \frac{d}{dx}(\sin x) = \cos x$

2) Differentiate cos x

Let $y = \cos x$

$y + \delta y = \cos(x+\delta x)$; $\delta y = \cos(x+\delta x) - \cos x = -2\sin\left(\frac{x+\delta x+x}{2}\right)\sin\left(\frac{x+\delta x-x}{2}\right)$

$\frac{\delta y}{\delta x} = \frac{-2\sin\left(\frac{x+\delta x+x}{2}\right)\sin\left(\frac{x+\delta x-x}{2}\right)}{\delta x}$;

The limit as $\delta x \to 0$; $\frac{\delta y}{\delta x} \to \frac{dy}{dx}$, $\sin\left(\frac{\delta x}{2}\right) \approx \frac{\delta x}{2} \therefore \frac{dy}{dx} = \frac{-\sin x \cdot \frac{\delta x}{2}}{{}^{1}/{}_{2}\delta x} = -\sin x$;

$\therefore \frac{d}{dx}(\cos x) = -\sin x$

3) Differentiate tan x

Let y=tan x ; $y + \delta y = \tan(x+\delta x)$; $\delta y = \tan(x+\delta x) - \tan x = \frac{\sin(x+\delta x)}{\cos(x+\delta x)} - \frac{\sin x}{\cos x}$

$$\delta y = \frac{\sin(x+\delta x)\cos x - \sin x\cos(x+\delta x)}{\cos x\cos(x+\delta x)} = \frac{\sin(x+\delta x - x)}{\cos x\cos(x+\delta x)} = \frac{\sin\delta x}{\cos x\cos(x+\delta x)}$$

$\frac{\delta y}{\delta x} = \frac{\sin\delta x}{\cos x\cos(x+\delta x)} \div \delta x$, limit as $\to 0$, $\frac{\delta y}{\delta x} \to \frac{dy}{dx}$; $\sin\delta x \to \delta x, \cos(x+\delta x) \to \cos x$

$\frac{dy}{dx} = \frac{\delta x}{\cos x\cos x}\cdot\frac{1}{\delta x} = \frac{1}{\cos^2 x} = \sec^2 x$;

$\therefore \frac{d}{dx}(\tan x) = \sec^2 x$

In summary

i. $\frac{d}{dx}(\sin x) = \cos x$; ii. $\frac{d}{dx}(\cos x) = -\sin x$; iii. $\frac{d}{dx}(\tan x) = \sec^2 x$

iv. $\frac{d}{dx}(\sec x) = \sec x\tan x$ v. $\frac{d}{dx}(\mathrm{cosec}\, x) = -\mathrm{cosec}\, x\cot x$ vi. $\frac{d}{dx}(\cot x) = -\mathrm{cosec}^2 x$

Exercise 2

(a)Differentiate the following with respect to (w. r. t) x

i. $y = \sin 4x$(Ans: 4cos 4x); ii. $y = \tan(x^2+1)^3$ iii. $y = \cos^3 x$ iv. $y = \sqrt{\sec 2x}$

v. $y = (x^2+1)\cot x$ vi. $y = \sec x\tan 2x$

(b) $y = \frac{\cos 2x}{\sin 3x}$

Derivatives of the inverse functions

Example 3

Find the derivatives with respect to x;

a) $\sin^{-1} x$

b)$\cot^{-1}(x^2 + 3)$

Solution

a) Let $y = \sin^{-1} x$; so $\sin y = x$;

Therefore, $\cos y \,.\frac{dy}{dx} = 1$;

$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{(1-\sin^2 y)}}$

$= \frac{1}{\sqrt{(1-x^2)}}$

b) Let $y = \cot^{-1}(x^2 + 3)$; so $\cot y = x^2 + 3$;

Therefore, $-cosec^2\, x.\frac{dy}{dx} = 2x$;

$\frac{dy}{dx} = -2x.\sin^2 x$

Exercise3

Find the derivatives with respect to x

a) $\cos^{-1} 3x$

$b)$ $\sin^{-1}(x^3 + 4x)$

$c)$ $4x^3 \tan^{-1}(2x)$

Topic: 13: Exponential and Logarithmic Functions

Learning Outcomes

The learner should be able to:

i) identify exponential functions.
ii) sketch smooth curves for exponential functions.

Sub-topic 1: Exponential Functions

Lesson: Identifying exponential functions

Introduction

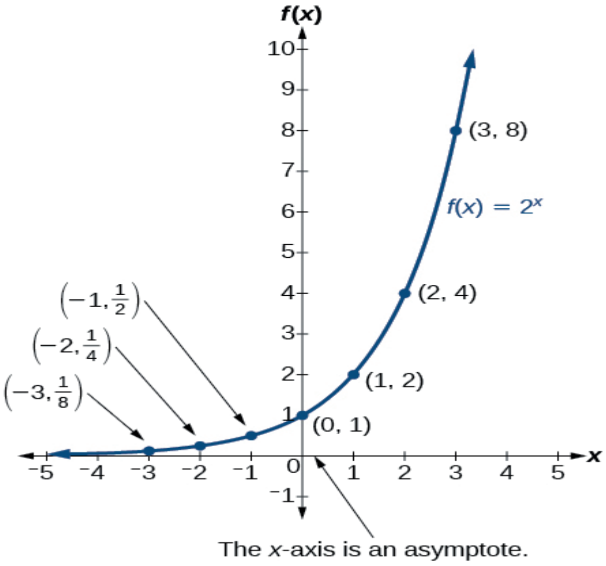
The word exponent is often used instead of index, and functions in which the variable is in the index (such as $2^x, 10^{\sin x}$) are called **exponential functions**.
Observe the table of values for a function of the form $f(x) = b^x$ whose base is greater than one. We'll use the function$f(x) = 2^x$. Observe how the output values in Table change as the input increases by 1.

x	-3	-2	-1	0	1	2	3
$f(x) = 2^x$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8

Notice from the table that

- The output values are positive for all values of:
- As increases, the output values grow smaller, approaching zero;
- Andas decreases, the output values grow without bound.

Each output value is the product of the previous output and the base, 2.



Note that the graph gets close to the x-axis, but never touches it.
The domain of $f(x)=2^x$ is all real numbers, the range is $(0,\infty)$, and the horizontal asymptote is $y=0$

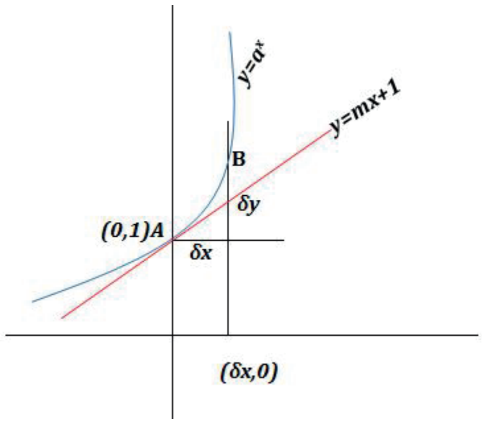
We call the **base 2** the ***constant ratio***. In fact, for any exponential function with the form $f(x)=ab^x$, b is the constant ratio of the function. This means that as the input increases by 1, the output value will be the product of the base and the previous output, regardless of the value of a.

Exercise 1

Sketch smooth curves of the following exponential functions for values of $x\ from\ -3\ to\ +3$

i) $f(x) = 0.25^x$ State the domain, range, and asymptote.
ii) $f(x) = 4^x$ State the domain, range, and asymptote.
iii) $f(x) = 2^x + 3$ State the domain, range, and asymptote.

The gradient of $y = a^x$ at $(0, 1)$



The gradient at AB is $\frac{\delta y}{\delta x} = \frac{a^{\delta x}-1}{\delta x}$;

Now as $\delta x \to 0$, the gradient of $AB \to m$.

It follows that the limit, as $\delta x \to 0$ of $\frac{a^{\delta x}-1}{\delta x}$ is m, the gradient of $y = a^x\ at\ (0,1)$

The form of $\frac{d}{dx}(a^x)$

The gradient of $y = a^x$ at any point $P(x, y)$ on the curve with the usual notation if θ is the point $(x + \delta x, y + \delta y)$

$$y + \delta y = a^{x+\delta x}; \qquad \delta y = a^{x+\delta x} - a^x = a^x(a^{\delta x} - 1)$$

Therefore, the gradient of PQ, $\frac{\delta y}{\delta x} = a^x\left(\frac{a^{\delta x}-1}{\delta x}\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(i)$

Now as $\delta x \to 0$, $\left(\frac{a^{\delta x}-1}{\delta x}\right) \to m.$; Then the RHS of (i) $\frac{\delta y}{\delta x} \to ma^x$; $\frac{dy}{dx} = ma^x$;

$$\textbf{Thus } \frac{d(a^x)}{dx} = ma^x$$

The exponential function e^x

Definition: e is the number such that the gradient $y = e^x$ at $(0,1)$ is **1.** e^x is called the exponential function.

$Thus\ \frac{d}{dx}(e^x) = e^x or\ y = e^x; \frac{dy}{dx} = y;$

Also $\int e^x\, dx = e^x + c$

Example 1

Find $\frac{dy}{dx}\ when\ y = e^{3x}$;

Let $u = 3x^2\ and \frac{du}{dx} = 6x;$

$then\ y = e^u\ and \frac{dy}{du} = e^u$

By chain rule, $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$= e^u.6x = 6xe^{3x^2}$

Sub-topic 2: Logarithmic Function

Learning Outcomes

The learner should be able to:

i) identify a logarithmic function.
ii) state the properties of logarithmic functions.
iii) find derivative of natural logarithm.
iv) apply the natural logarithm to differentiate exponential functions.

Lesson: Further theory of logarithms

Definition

The logarithm of ***b*** to base ***a*** written $log_a b$, is the power to which the base must be raised to equal to b.

Thus since $10^2 = 100, 2 = \log_{10} 100$. ; $If\ a^x = b\ , x = \log_a \text{b}$

Basic rules

i) $\log_c(ab) = log_c a + log_c b$

ii) $\log_c\left(\frac{a}{b}\right) = log_c a - log_c b$

iii) $\log_c a^n = n\ log_c a$

Natural logarithms

Logarithms to the base e are called natural logarithms or Nipierian logarithms., $\log_e x = In\ x$

Exercise 2.

1. Express as a single logarithm:
   a) $2 \log a - 2 + \log 2a$,
   b) $3 \log_e x + 3 - \log_e 3x$
   c) $\frac{1}{2}\log_e(1 + y) + \frac{1}{2}\log_e(1 - y) + \log_e k$
2. Express in terms of $\log_c a$
   a) $\log_e 3a$ b) $\log_e\left(\frac{a}{3}\right)$ c) $\log_e\left(\frac{1}{3}a^{-1}\right)$
3. Solve the equations; a) $2^{\left(\frac{2}{x}\right)} = 32$ b) $3^{x+1} = 12$; c) $\frac{3}{2}\log_{10}a^3 - \log_{10}\sqrt{a} - 2\log_{10}a = 4$
   a) $\log_{10}y - 4\log_y 10 = 0$

Derivative of In

Find $\frac{d}{dx}(In\ x)$,

we write $y = In\ x\ as\ x = e^y$

Differentiate both sides with respect to x.

$\frac{d}{dx}(x) = \frac{d}{dy}(e^y) \times \frac{dy}{dx}$;

$1 = e^y\frac{dy}{dx}$;

$\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$;

$\therefore \frac{d}{dx}(In\ x) = \frac{1}{x}$;

Derivative of Inf(x)

Find $\frac{d}{dx}(In\ f(x))$,

we write $y = Inf(\ x)\ as\ f(x) = e^y$

Differentiate both sides with respect to x.

$\frac{d}{dx}(f(x)) = \frac{d}{dy}(e^y) \times \frac{dy}{dx}$;

$f^{I}(x) = e^{y}\frac{dy}{dx};$

$\frac{dy}{dx} = \frac{f^{I}(x)}{e^{y}} = \frac{f^{I}(x)}{f(x)};$

$\therefore \frac{d}{dx}(In\, f(x)) = \frac{f^{I}(x)}{f(x)};$

Then $\int \frac{f'(x)}{f(x)}dx = \ln f(x) + C;$

It follows that $\int \frac{1}{x}dx = In\,(kx)\ where\ c = In\ k$

Example 3

Differentiate

(a) ln (2x)

(b) ln(x^2 + 3x)

Solution.

(a) $\frac{d(\ln(2x))}{dx} = \frac{2}{2x} = \frac{1}{x}$

(b) $\frac{d(\ln(x^2+3x))}{dx} = \frac{2x+3}{x^2+3x}$

Example 4

Find.

(a) $\int \frac{2x+1}{x^2+x-1}dx$

(b) $\int \frac{cosx}{sinx}dx$

Solution

(a) $\int \frac{2x+1}{x^2+x-1}dx$

You realise that $\frac{d(x^2+x-1)}{dx} = 2x + 1$ which is the numerator.

Therefore, $\int \frac{2x+1}{x^2+x-1}dx = ln\,(x^2 + x - 1) + c$

(b) $\int \frac{cosx}{sinx}dx$

You realise that $\frac{d(sinx)}{dx} = cosx$ which is the numerator.

Therefore, $\int \frac{cosx}{sinx}dx = ln\,(sinx) + c$

(c) $\int \frac{1}{2x}dx;$

$\frac{d}{dx}2x = 2$ Which is twice the numerator.

$\therefore \int \frac{1}{2x}dx = \frac{1}{2}\int \frac{1}{x}dx = \frac{1}{2}\ In\ x + c$

(d) $\int \frac{x}{x^2+1}dx$

You realise that $\frac{d}{dx}(x^2 + 1) = 2x$ which is twice the numerator.

$\therefore \int \frac{x}{x^2+1}dx = \frac{1}{2}\ In(x^2 + 1) + c$

Derivative of a^x

$\frac{d}{dx}(a^x) = ma^x$

$\therefore \frac{d}{dx}(a^x) = a^x \, In \, a;$

Also, $\int a^x \, dx = \frac{a^x}{In \, a} + c$

Exercise 3.

1. Differentiate the following:-
   - (a) e^{2x}
   - (b) e^{4x-1}
   - (c) e^{x^2+x}
   - (d) e^{sinx}
   - (e) e^{tanx}
   - (f) $e^{\sqrt{x}}$
2. Integrate the following:-
   - (a) e^x
   - (b) $4e^{4x}$
   - (c) $2xe^{x^2}$
   - (d) $cosxe^{sinx}$
   - (e) $(2x+2)e^{x^2+2x}$
   - (f) e^{3x}
3. Differentiate the following:-
   - (a) $\ln(3x)$
   - (b) $\ln(x^2 - 2x)$
   - (c) $\ln(sinx)$
   - (d) $\ln(\sqrt{x})$
4. Find:-
   - (a) $\int \frac{2x-1}{x^2-x-1} dx$
   - (b) $\int -\frac{sinx}{cosx} dx$
   - (c) $\int \frac{1}{2x-1} dx$
   - (d) $\int \frac{4x+4}{x^2+2x+2} dx$
5. Differentiate the following:-
   - (a) 2^x
   - (b) 3^{2x}
   - (c) 5^{x^2}
   - (d) 4^{sinx}
6. Integrate the following:-
   - (a) 3^x
   - (b) 2^{3x-1}
   - (c) $x4^{x^2}$
   - (d) $sec^2x \, 5^{tanx}$
7. If $y = Ae^{-x}\cos(x+\alpha)$where A and α are constants, prove that
   - i) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 0$
   - ii) $\frac{d^4y}{dx^4} + 4y = 0$

Topic: 14: Maclaurin's Theorem

Learning Outcomes

The learner should be able to:

i) relate Maclaurin's theorem to the binomial expansion.

ii) apply Maclaurin's theorem in expansions for approximations.

Sub-topic: Maclaurin's Theorem

Lesson: Maclaurin's theorem

Bearing in mind the relationship $x = a + h$, where a is a constant, and x and h are variable, we see that there is a special case given by $a = 0$ when $x = h$ and either form of Taylor's theorem given it reduces to

$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{iv}(0)}{4!}x^4 + \cdots$. This is a form of **Maclaurin's** theorem.

Example 1:

Use Maclaurin's theorem to expand in $In(1 + x)$ in ascending powers of x as far as the term in x^5

$$f(x) = In(1+x), \qquad f(0) = 0$$
$$f'(x) = (1+x)^{-1}, \qquad f'(0) = 1$$
$$f''(x) = (1+x)^{-2}, \qquad f''(0) = -1$$
$$f'''(x) = 2(1+x)^{-3}, \qquad f'''(0) = 2!$$
$$f^{iv}(x) = -3 \times 2(1+x)^{-4}, \qquad f^{iv}(0) = -3!$$
$$f^{v}(x) = 4 \times 3 \times 2(1+x)^{-5}, \quad f^{v}(0) = 4!$$

By Maclaurin's theorem

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{iv}(0)}{4!}x^4 + \frac{f^{v}(0)}{5!}x^5 + \cdots$$
$$In(1+x) = 0 + 1x + \frac{-1}{2!}x^2 + \frac{2!}{3!}x^3 + \frac{-3!}{4!}x^4 + \frac{4!}{5!}x^5 + \cdots; \therefore In(1+x)$$
$$= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 - \cdots$$

Exercise.

1. Use maclaurin's theorem
   a) To expand e^x in ascending powers of x as far as the x^5 term
   b) To show that when x is small, $\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!}$ and $find$ $\sin 17^0 11'$.
   c) To find the first three terms of $\cos x$.

2. (a) Write down the first three terms in $\frac{1}{2} In\left(\frac{1+x}{1-x}\right)$

(b) Hence solve

(i) $In\ 3$ (ii) $In\left(\frac{5}{3}\right)$

3. Apply Maclaurin's theorem to expand each of the following upto the term indicated in the corresponding bracket.

i) $\tan^{-1} x$ (x^3)

ii) 2^x (x^3)

iii) $e^x \cos x$ (x^5)

iv) $In\left(x + \sqrt{(x^2+1)}\right)$ (x^3)

Topic 15: Integration II

Sub-topic 1: Function and its Derivative

Learning Outcomes

The learner should be able to:

i) recognise a function and its derivative and integrate.

Lesson: Recognizing the presence of a function and its derivative.

Example 1 :

Find $\int x(3x^2+2)^4\,dx$

[We note that the outside the bracket is a constant × the derivative of the expression inside the bracket. We deduce that the integral is a function of $(3x^2+2)$]

$$\frac{d}{dx}(3x^2+2)^5 = 5(3x^2+2)^4 \times 6x$$
$$= 30x(3x^2+2)^4;$$
$$\therefore \frac{d}{dx}\left(\frac{1}{30}(3x^2+2)^5\right) = x(3x^2+2)^4$$

Hence $\int x(3x^2+2)^4\,dx = \frac{1}{30}(3x^2+2)^5 + c$

Or Let $u = 3x^2+2$; $du = 6xdx$; $\frac{du}{6x} = dx$;

$$\int x(3x^2+2)^4\,dx = \int x.u^5.\frac{du}{6x}$$
$$= \frac{u^5}{30} + c$$
$$= \frac{1}{30}(3x^2+2)^5 + c$$

Example 2

Find $\int sin\,^2 4x\,cos\,4x\,dx$ [We note that $cos\,4x$ is a constant × the derivative of $sin\,4x$]

So $\frac{d}{dx}(\sin^3 4x) = 3(\sin 4x)^2 \times cos\,4x \times 4 = 12\sin^2 4x\cos 4x$

Hence $\int \sin^2 4x\cos 4x\;dx = \frac{1}{12}\sin^3 4x + c.$

Exercise 1

Find:-

(a) $\int (x+2)^{10}\,dx$

(b) $\int x(x^2+2)^3\,dx$

(c) $\int sin\,^2 x\,cos\,x\,dx$

Sub-topic 2: Change of Variables

Learning Outcomes

The learner should be able to integrate by change of variable.

Lesson: Changing the variable(substitution)

Example 3

Find $\int x(3x^2+2)^4\,dx$.

$let\ u = 3x^2+2$; Then $\frac{dy}{dx} = x(3x^2+2)^4$

If $= 3x^2+2$; Then by the chain rule $\frac{dy}{du} = \frac{dy}{dx} \times \frac{dx}{du}$;

$\frac{dy}{du} = x(3x^2+2)^4\frac{dx}{du}$;

Integrating with respect to u.

$y = \int x(3x^2+2)^4\frac{dx}{du}du$;

$But\ u = 3x^2+2; \quad \therefore \frac{du}{dx} = 6x\ and\ \frac{dx}{du} = \frac{1}{6x}$

$$\int x(3x^2+2)^4\,dx = \int x(3x^2+2)^4\frac{1}{6x}du$$
$$= \int \frac{1}{6}u^4du = \frac{1}{30}u^5$$
$$= \frac{1}{30}(3x^2+2)^5 + \text{c}$$

So for $y = \int f(x)\,dx$; Then $\frac{dy}{dx} = f(x)$;

From chain rule, $\frac{dy}{du} = \frac{dy}{dx} \times \frac{dx}{du}$; $\therefore \frac{dy}{dx} = f(x) \times \frac{dx}{du}$

$$\therefore \boldsymbol{\int f(x)\,dx = \int f(x)\frac{dx}{du}\,du}$$

This is the formula for substitution or change of variable.

Exercise 2.

1. Integrate each of the following using the substitution(change of variable) indicated.
   (a) $\int \sin^2 4x \cos 4x\,dx, put\ u = \sin 4x$
   (b) $\int \sin^5 x\,dx\,, put\ u = \cos x$
   (c) $\int x\sqrt{(3x-1)}dx\,, put\ u = \sqrt{(3x-1)}$
   (d) $\int x\sqrt{(2x+1)}dx\,, put\ u = \sqrt{(2x+1)}$
   (e) $\int x\sqrt{(2x+1)}dx\,, put\ u = 2x+1$
   (f) $\int x\,(3x-2)^6dx, put\ u = 3x-2$
2. Prove that.
   a) $\int \sin x\sqrt{\cos x}\,dx = -\frac{2}{3}(\cos x)^{\frac{3}{2}} + c$

b) $\int \cot^2 x\, cosec^2\, x\, dx = -\frac{1}{3}\cot^3 x + c$

c) $\int \frac{\sin\sqrt{x}}{\sqrt{x}}\, dx = -2\cos\sqrt{x} + c$

d) $\int \left(\frac{x-1}{\sqrt{(2x+3)}}\right) dx = \frac{1}{6}(2x+3)^{\frac{3}{2}} - 3\sqrt{(2x+3)} + c$

Definite integrals and changing the limits

Example 4.

Evaluate $\int_{\frac{1}{2}}^{3} x\sqrt{(2x+3)}dx = 11.6$

$\int_{\frac{1}{2}}^{3} x\sqrt{(2x+3)}dx$

$let\ u = (2x+3)^{\frac{1}{2}} \Rightarrow u^2 = 2x+3\ and\ x = \frac{u^2-3}{2} \qquad \therefore \frac{dx}{du} = 2$

We have to change the limits.

x	u^2	u
$\frac{1}{2}$	4	2
3	9	3

$\int f(x)\,dx = \int f(x)\frac{dx}{du}\,du$

$= \int_2^3 \left(\frac{u^2-3}{2}\right) \times 2\,du$

$= \int_2^3 u^2 - 3du$

$= \left[\frac{u^3}{3} - 3u\right]_2^3$

$= \left(\frac{3^3}{3} - 3\times 3\right) - \left(\frac{2^3}{3} - 3\times 2\right) = 0 - (-3.3333)$

= 3.3333

Exercise 3

Prove that:

(a) $\int_2^3 \frac{x}{\sqrt{(x^2-3)}}dx = \sqrt{6} - 1$

(b) $\int_0^{\frac{\pi}{4}} \cos^3 x \sin x\, dx = \frac{3}{16}$

Sub-topic 3: Pythagoras' theorem, odd powers of $\sin x$, $\cos x$ etc.

Pythagorean identities are in the forms.

$$cos^2 x + sin^2 x = 1,\ cot^2 x + 1 = cosec^2 x, 1 + tan^2 x = sec^2 x$$

Example 5.

Find $\int \boldsymbol{sin\,^5 x\,dx}$

$\int \boldsymbol{sin\,^5 x\,dx} = \int \boldsymbol{sin\,^4 x\,sin\,x\,dx}$

$= \int (\mathbf{1} - cos^2 x)^2 \sin x\,dx$

$= \int (\sin x - 2\cos^2 x \sin x + \cos^4 x\ \ sinx)dx;$

$\therefore \int \boldsymbol{sin\,^5 x\,dx = -cos\,x + \frac{2}{3}cos^3 x - \frac{1}{5}cos^5 x + c}$

Exercise 4

Find:-

(a) $\int sin\,^3 x\,dx$

(b) $\int cos\,^3 2x\,dx$

(c) $\int cos\,^5 x\,dx$

Sub-topic 4: Even Powers

Learning Outcome

The learner should be able to use double angle formulae for integrating even powers of cosine and sine.

Lesson:

Even powers of $\boldsymbol{sin\,x, cos\,x}$.

Two very important formulae derived from the double-angle formulae are

$$cos^2 x = \frac{1}{2}(1 + \cos 2x)\ and\ sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

Example 6.

1.Find $\int sin\,^2 2x\,dx$

But $sin^2 2x = \frac{1}{2}(1 - \cos 4x)$ Note: double the existing angle.

Therefore, $\int sin\,^2 2x\,dx = \int \frac{1}{2}(1 - \cos 4x)\,dx$

$= \int \frac{1}{2} - \frac{1}{2}cos4x\,dx$

$= \frac{1}{2}x - \frac{1}{2}\left(\frac{1}{4}sin4x\right) + c$

$= \frac{1}{2}x - \frac{1}{8}sin4x + c$

2. Find $\int cos^4 x\,dx$

But $cos^2 x = \frac{1}{2}(1 + \cos 2x)$ Note: double the existing angle.

Therefore, $\int (cos^2 x)^{\ 2}\, dx = \int \left(\frac{1}{2}(1 - \cos 4x)\right)^2 dx$

$$= \frac{1}{4}\int (1 - 2cos2x + cos^2\, 2x)dx$$
$$= \frac{1}{4}\left(x - 2\left(\frac{1}{2}sin2x\right) + \int \frac{1}{2}(1 + cos4x\right)$$
$$= \frac{1}{4}x - \frac{1}{4}sin2x + \frac{1}{8}x + \frac{1}{8}\left(-\frac{1}{4}sin4x\right)$$
$$= \frac{3}{8}x - \frac{1}{4}sin2x - \frac{1}{32}sin4x + c$$

Exercise 5.

Find:-

(a) $\int sin^4 x\, dx$

(b) $\int cos^2 3x\, dx$

Sub-topic 4: Inverse Trigonometrical Functions

Learning Outcome

The learner should be able to integrate functions of the form:

i) $\dfrac{1}{\sqrt{(a^2 - b^2x^2)}}$

ii) $\dfrac{1}{a^2 + b^2x^2}$

iii) $\int \dfrac{f'(x)}{f(x)} dx = \ln f(x) + c$

Lesson: Integration using the inverse trigonometrical functions

The inverse trigonometrical functions were introduced in derivatives of trigonometry, you are advised to revise it before proceeding further. The following angles lie between 0 and 90⁰· inclusive. Express them in degrees and radians in terms of π. A)$\tan^{-1} 1$ B) $\sin^{-1}\frac{1}{2}$ C)$\frac{1}{2}\cos^{-1} 1$ D) $\frac{2}{3}\cot^{-1} 1$ E)$2\,\text{cosec}^{-1}\sqrt{2}$

The inverse sine function may be written as $arcsin\, x$ or as $\sin^{-1} x$. Both forms are in current use and both will be used. The expression $\sqrt{(1-x^2)}$ may be reduced to a rational form by changing variable to u, where $x = sin\, u$, thus

$\sqrt{(1-x^2)} = \sqrt{(1-sin^2 u)} = \sqrt{\cos^2 u} = \cos u$; This is used in the following example

Example 7

1 $Find \int \frac{1}{\sqrt{(1-4x^2)}} dx$

$$\int \frac{1}{\sqrt{(1-4x^2)}} dx$$

$$= \int \frac{1}{\sqrt{(1-4x^2)}} \frac{dx}{du}\, du,$$

let $4x^2 = \sin^2 x \;\; \therefore 2x = \sin u\,; \;\; \Rightarrow \;\; \frac{dx}{du} = \frac{1}{2}\cos u$ and $u = \sin^{-1} 2\,x$

$$= \int \frac{1}{\sqrt{(1-sin^2 u)}} \frac{1}{2}\cos u\;\, du$$

$$= \frac{1}{2}\int \frac{1}{\cos u}\cos u\;\, du = \frac{1}{2}\int dx$$

$$= \frac{1}{2}u + c = \sin^{-1} 2x + c$$

2. Find $\int \frac{1}{\sqrt{(9-4x^2)}} dx;$

$$\int \frac{1}{\sqrt{(9-4x^2)}} dx$$

$$= \int \frac{1}{\sqrt{(9-4x^2)}} \frac{dx}{du}\, du\,,$$

let $4x^2 = 9\sin^2 x \ \therefore 2x = 3\sin u\,;\ \Rightarrow \ \frac{dx}{du} = \frac{3}{2}\cos u$ and $u = \sin^{-1}\frac{2x}{3}$

$$= \int \frac{1}{\sqrt{(9-9sin^2 u)}} \frac{3}{2}\cos u\ du$$
$$= \int \frac{1}{3\sqrt{(1-sin^2 u)}} \frac{3}{2}\cos u\ du$$
$$= \frac{1}{2}\int \frac{1}{\cos u}\cos u\ du$$
$$= \frac{1}{2}\int dx$$
$$= \frac{1}{2}u + c = \frac{1}{2}\sin^{-1}\frac{2x}{3} + c.$$

3. Find

$$\int_0^1 \left(\frac{1}{\sqrt{(9-4x^2)}}\right) dx$$

$4x^2 = 9\sin^2 u\,, 2x = 3\sin u\ \ and\ x = \frac{3}{2}\cos u\, du$; And then $\sin u = \frac{2x}{3};\ u = \sin^{-1}\left(\frac{2x}{3}\right)$

Limits

x	0	1
u	0	0.7297 rad

$$\therefore \int_0^1 \left(\frac{1}{\sqrt{(9-4x^2)}}\right) dx = \int_0^{0.7297} \left(\frac{1}{\sqrt{9(1-\sin^2 u)}}\right)\frac{3}{2}\cos u\, du = \int_0^{0.7297} \frac{1}{2}\, du = \left[\frac{1}{2}u\right]_0^{0.7297}$$
$$= 0.365, correct\ to\ three\ significant\ figures.$$

Exercise 6.

Find.

1. $\int \frac{1}{25-9x^2}dx$
2. $\int \frac{1}{\sqrt{(4-(x-1)^2)}}dx$
3. $\int \frac{1}{\sqrt{(1+8x-4x^2)}}dx$(clue : complete squares)
4. $\int \frac{2x+3}{\sqrt{(4-x^2)}}dx$

For $\int \frac{1}{a^2+b^2x^2}dx$; let $a^2 + b^2x^2 = a^2 + a^2tan^2u = a^2sec^2u$, i.e. $bx = atan\,u$

Example 8.

Find $\int \frac{1}{25+4x^2}dx$

$25 + 4x^2 = 25 + 25tan^2u = 25sec^2u,$

So let $4x^2 = 25\tan^2 u \ \therefore 2x = 5\tan u\,;\ \Rightarrow \ \frac{dx}{du} = \frac{5}{2}sec^2u$ and $u = \tan^{-1}\frac{2x}{5}$

$.\int \frac{1}{25+4x^2}dx$

$$= \int \frac{1}{25+25tan^2u} \times \frac{5}{2}sec^2u\, du$$

$$= \int \frac{1}{25(1+tan^2u)} \times \frac{5}{2} sec\ ^2u\, du$$
$$= \int \frac{1}{25sec\ ^2u} \times \frac{5}{2} sec\ ^2u\, du$$
$$= \int \frac{1}{10} du = \frac{1}{10} u + c$$
$$= \frac{1}{10} \tan^{-1} \frac{2x}{5} + c$$

Exercise 7

1. Find:-

(a) $\int \frac{1}{1+9x^2} dx$

(b) $\int \frac{1}{(x+2)^2+9} dx$

(c) $\int \frac{1}{x^2-2x+5} dx$

2. Prove that $\int \frac{1}{(x+3)^2+25} dx = \frac{1}{5} \tan^{-1}\left(\frac{x+3}{5}\right) + c$

1

For $\int \frac{f'(x)}{f(x)} dx = In\, f(x) + c$ refer to the derivative of natural logarithms

Sub-topic 5: Partial Fractions

Learning Outcome

The learner should be able to integrate partial fractions.

Lesson: Integration

Introduction

Early training in algebra teaches us how to simplify an expression such as $\frac{1}{x-1} - \frac{1}{x+1}$ by reducing it to $\frac{2}{x^2-1}$. Given a fraction such as $\frac{5}{x^2+x-6}$ whose denominator factorises, we may split it up into its component fractions, writing it as $\frac{1}{x-2} - \frac{1}{x+3}$; it is now said to be in **partial fraction**s, this enables us to find $\int \frac{5}{(x-2)(x+3)} dx$ as it stands, by using partial fractions.(Refer to senior five work on partial fractions of term one).

$$\int \frac{5}{(x-2)(x+3)} dx = \int \left\{\frac{1}{x-2} - \frac{1}{x+3}\right\} dx = In(x-2) - In(x+3) + c = In\left\{\frac{k(x-2)}{x+3}\right\}$$

Example 9:

Find $\int \frac{(2x-1)}{(x+1)^2} dx$; Let $\frac{(2x-1)}{(x+1)^2} \equiv \frac{A}{x+1} + \frac{B}{(x+1)^2}$;

we find that $A = 2$ and $B = -3$

$$\therefore \int \frac{(2x-1)}{(x+1)^2} dx \equiv \int \left\{\frac{2}{x+1} - \frac{3}{(x+1)^2}\right\} dx$$
$$= \int \left\{\frac{2}{x+1}\right\} dx - \int \left\{\frac{3}{(x+1)^2}\right\} dx$$
$$= 2\, In(x+1) + 3(x+1)^{-1} + C$$

Example 10

Evaluate $\int_2^3 \frac{5+x}{(1-x)(5+x^2)} dx$ correct to three significant figures.

Let $\frac{5+x}{(1-x)(5+x^2)} = \frac{A}{1-x} + \frac{Bx+C}{(5+x^2)}$ we find that $A = 1, B = 1, C = 0$

$$\int_2^3 \frac{5+x}{(1-x)(5+x^2)} dx$$

$$= \int_2^3 \left(\frac{1}{1-x} + \frac{x}{(5+x^2)}\right) dx = \left[-In(x-1) + \frac{1}{2}In(5+x^2)\right]_2^3$$

$$= \left(-In2 + \frac{1}{2}In14\right) — In1 + \frac{1}{2}In9$$

$$= \frac{1}{2}In14 - In2 - In3 = -0.472$$

Exercise 8.

Find:-

a) $\int \frac{1}{x^2-9} dx$

b) $\int \frac{2x+2}{(2x-3)^2} dx$

Sub-topic 6: Integration by Parts

Learning Outcome

The learner should be able to integrate by parts.

Lesson: Integration by parts

If U and V are two functions of x, $\frac{d(UV)}{dx} = V\frac{dU}{dx} + U\frac{dV}{dx}$; Integrating each side with respect to x

$UV = \int V\frac{du}{dx}dx + \int U\frac{dv}{dx}dx$;

$$\therefore \int \boldsymbol{U\frac{dv}{dx}dx = UV - \int V\frac{du}{dx}dx}$$

There is no formula for choosing U and $\boldsymbol{\frac{du}{dx}}$. But we can try to group some types of integrands.

Note: There are others that may be isolated cases. So you will need to choose carefully.

The forms $\int \boldsymbol{x^n cosaxdx}$, $\int \boldsymbol{x^n e^{ax} dx}, \int \boldsymbol{x^n(ax+b)^m dx}$.

Here we differentiate the x^nterm by making it U.

Example 11.

1. Find $\int x^2 cos\,x\,dx$.

Let $U = x^2 \Rightarrow \frac{dU}{dx} = 2x$ and $\frac{dV}{dx} = cosx \Rightarrow V = sinx$

By using $\int \boldsymbol{U\frac{dv}{dx}dx = UV - \int V\frac{du}{dx}dx}$

$$\int x^2 cos\,x\,dx = x^2 sinx - \int 2xsinx\,dx$$

Also, $\int 2xsinx\,dx$.

Let $U = 2x \Rightarrow \frac{dU}{dx} = 2x$ and $\frac{dV}{dx} = sinx \Rightarrow V = -cosx$

Again by parts, $\int 2xsinx\,dx = 2x \times -cosx - \int 2(-cosx)\,dx$

$= -2xcosx + 2sinx$

Therefore, $\int x^2 cos\,x\,dx = x^2 sinx - (-2xcosx + 2sinx)$

$= x^2 sinx + 2xcosx - 2sinx + c.$

2. Find $\int x^2 e^{2x}\,dx$.

Let $U = x^2 \Rightarrow \frac{dU}{dx} = 2x$ and $\frac{dV}{dx} = e^{2x} \Rightarrow V = \frac{1}{2}e^{2x}$

By using $\int \boldsymbol{U\frac{dv}{dx}dx = UV - \int V\frac{du}{dx}dx}$

$\int x^2\,e^{2x} dx = x^2 \times \frac{1}{2}e^{2x} - \int 2x \times \frac{1}{2}e^{2x}\,dx$

Also, $\int xe^{2x}\,dx$.

Let $U = x \Rightarrow \frac{dU}{dx} = 1$ and $\frac{dV}{dx} = e^{2x} \Rightarrow V = \frac{1}{2}e^{2x}$

Again by parts, $\int xe^{2x}\,dx = x \times \frac{1}{2}e^{2x} - \int \frac{1}{2}e^{2x}\,dx$

$= \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x}$

Therefore, $\int x^2\,e^{2x} dx = \frac{1}{2}x^2 e^{2x} - \left(\frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x}\right)$+c

Exercise 9.

Find

(a) $\int x^2(x+1)^6 dx$

(b) $\int x^2 e^x\,dx$

(c) $\int x^2 sin\,x\,dx$

The forms $\int e^{ax} sin\,bx\,dx$, $\int e^{ax} sin\,bx\,dx$, $\int \frac{tan^{-1}x}{1+x^2}dx$, $\int \frac{sin^{-1}x}{\sqrt{1-x^2}}dx$, $\int \frac{cos^{-1}x}{\sqrt{1-x^2}}dx$

We let the given integration be I.

Then we make I the subject.

Example 12

1.Find $\int e^x sin\,x\,dx$

Let $U = e^x \Rightarrow \frac{dU}{dx} = e^x$ and $\frac{dV}{dx} = sinx \Rightarrow V = -cosx$

By using $\int \boldsymbol{U\frac{dv}{dx}dx = UV - \int V\frac{du}{dx}dx}$

$\mathrm{I} = -e^x cosx + \int e^x cosx\,dx$

Let $U = e^x \Rightarrow \frac{dU}{dx} = e^x$ and $\frac{dV}{dx} = cosx \Rightarrow V = sinx$

$\mathrm{I} = -e^x cosx + e^x sinx - \int e^x sin\,x\,dx$

$I = -e^x cosx + e^x sinx - I$

$2I = -e^x cosx + e^x sinx$

$I = \frac{1}{2}(e^x sinx - e^x cosx) + c$

2.Find $\int \frac{tan^{-1}x}{1+x^2}dx$

We let $I = \int \frac{tan^{-1}x}{1+x^2} dx = \int tan^{-1}x \times \frac{1}{1+x^2} dx$

$U = tan^{-1}x \Rightarrow \frac{dU}{dx} = \frac{1}{1+x^2}$ and $\frac{dV}{dx} = \frac{1}{1+x^2} \Rightarrow V = tan^{-1}x$

By using $\int \boldsymbol{U \frac{dv}{dx} dx = UV - \int V \frac{du}{dx} dx}$

$$\text{I} = tan^{-1}x \times tan^{-1}x - \int tan^{-1}x \times \frac{1}{1+x^2} dx$$

$$I = tan^{-1}x \times tan^{-1}x - I$$

$$2I = tan^{-1}x \times tan^{-1}x$$

There fore $I = \frac{1}{2}(tan^{-1}x)^2 + c$

Exercise 10.

Find:

(a) $\int e^x cos\ x\ dx$

(b) $\int \frac{sin^{-1}x}{\sqrt{1-x^2}} dx$

(a) $\int e^x sin\ 2x\ dx$

The forms $\int Inx\ dx, \int sin^{-1}x\ dx, \int cos^{-1}x\ dx$ and $\int tan^{-1}x\ dx$

We make the factor of one ' visible' and then integrate it by making it be $\frac{dv}{dx}$

Example 13.

Find $\int Inx\ dx$.

$\int Inx\ dx = \int 1 \times Inx\ dx$

$U = lnx \Rightarrow \frac{dU}{dx} = \frac{1}{x}$ and $\frac{dV}{dx} = 1 \Rightarrow V = x$

By using $\int \boldsymbol{U \frac{dv}{dx} dx = UV - \int V \frac{du}{dx} dx}$

$$\int Inx\ dx = (lnx)x - \int x \times \frac{1}{x} dx$$

$$= (lnx)x - \int dx$$

$$= (lnx)x - x + c$$

Exercise 11.

Find:

(a) $\int In2x\ dx$

(b) $\int sin^{-1}x\ dx$

(c) $\int tan^{-1}2x\ dx$

The forms $\int x^n(Inx)^m\ dx, \int x^n sin^{-1}x\ dx, \int x^n cos^{-1}x\ dx$ and $\int x^n tan^{-1}x\ dx$

We integrate the $\boldsymbol{x^n}$ factor by making it be $\frac{dv}{dx}$

Example 14.

Find $\int x^2(Inx)^2\ dx$.

$U = (Inx)^2 \Rightarrow \frac{dU}{dx} = 2lnx \times \frac{1}{x}$ and $\frac{dV}{dx} = x^2 \Rightarrow V = \frac{x^3}{3}$

By using $\int \boldsymbol{U\frac{dv}{dx}dx = UV - \int V\frac{du}{dx}dx}$

$\int x^2(Inx)^2\,dx = (lnx)^2 \times \frac{1}{3}x^3 - \int \frac{1}{3}x^3 \times 2lnx \times \frac{1}{x}dx$

$= \frac{x^3(lnx)^2}{3} - \frac{2}{3}\int x^2 lnxdx$

Again, we let $U = Inx \Rightarrow \frac{dU}{dx} = \frac{1}{x}$ and $\frac{dV}{dx} = x^2 \Rightarrow V = \frac{x^3}{3}$

$\int x^2(Inx)^2\,dx = \frac{x^3(lnx)^2}{3} - \frac{2}{3}\left(lnx \times \frac{x^3}{3} - \int \frac{x^3}{3} \times \frac{1}{x}dx\right)$

$= \frac{x^3(lnx)^2}{3} - \frac{2x^3 lnx}{9} + \frac{x^3}{27} + c$

Exercise 12.

1. Find:

(a) $\int x^2 lnx\,dx$

(b) $\int x^2 tan^{-1}x\,dx$

2. Prove that $\int xsin^{-1}x\,dx = \frac{1}{4}(2x^2 - 1)sin^{-1}x + \frac{1}{4}x\sqrt{(1 - x^2)} + C$

Sub-topic 7: tan substitutions.

Learning Outcome

The learner should be able to integrate by using tan substitution.

Lesson: The change of variable $t = tan\frac{x}{2}$

Of the trigonometrical ratios, sec x and $cosec\ x$ have not yet been integrated in this book.

If $t = tan\frac{x}{2}$, $\sin x = \frac{2t}{1+t^2}$, $\cos x = \frac{1-t^2}{1+t^2}$ *and* $\tan x = \frac{2t}{1-t^2}$

Example 15.

1.Find $\int cosec\ x\,dx$

$\int cosec\ x\,dx = \int \frac{1}{sinx}dx$

Let $t = tan\frac{x}{2} \Rightarrow \frac{dt}{dx} = \frac{1}{2}\sec^2\frac{x}{2} \Rightarrow \frac{dx}{dt} = \frac{2}{1+t^2}$

$= \int \frac{1}{\frac{2t}{1+t^2}}\frac{2}{1+t^2}dt$

$= \int \frac{1}{t}dt$

$= lnt$

$= In\tan\frac{x}{2} + c$

Exercise 13

1. Prove that $\int \sec x\,dx = In\,(\sec x + \tan x) + C$

2. (a)Find $\int \frac{1}{3+5\cos\frac{1}{2}x}dx$.   (Answer: $In\ k\left(\frac{\sqrt{(2+\tan\frac{1}{4}x)}}{\sqrt{(2-\tan\frac{1}{4}x)}}\right)$

(b) $\int \frac{1}{1+\sin 3\theta} d\theta$ (Answer: $-\frac{2}{3}\left(1+\tan\frac{3}{2}\theta\right)^{-\frac{1}{2}}+C$)

(c) $\int cosec\ 2x\, dx$ (Answer: $\frac{1}{2} In \tan x + C$)

The change of variable $t = \tan x$

An integrand containing $\sin x$ and $\cos x$, particularly even powers of these, may often be expressed as a function of $\tan x$ and $\sec x$. In such a case the change of variable $t = \tan x$

Example 16

1. Find $\int \frac{1}{1+\sin^2 x} dx$

Dividing thru by $\cos^2 x$; $\int \frac{1}{1+\sin^2 x} dx = \int \frac{\sec^2 x}{\sec^2 x + \tan^2 x} dx$,

Let $t = \tan x$; $\frac{dt}{dx} = \sec^2 x = 1 + t^2 \Rightarrow \frac{dx}{dt} = \frac{1}{1+t^2}$

$$\int \frac{\sec^2 x}{\sec^2 x+\tan^2 x} \cdot \frac{dx}{dt} . dt$$

$$= \int \left(\frac{1}{1+t^2}\right)^{-1} \cdot \frac{1}{1+2t^2} \cdot \frac{1}{1+t^2} dt$$

$$= \int \frac{1}{1+2t^2} dt;$$

Let $1 + 2t^2 = 1 + \tan^2 u = \sec^2 u$

$2t^2 = \tan^2 u$; $\sqrt{2}t = \tan u$; $\sqrt{2}\, dt = \sec^2 u\, du$; $dt = \frac{1}{\sqrt{2}} \sec^2 u\, du$

$$\int \frac{\sec^2 x}{\sec^2 x+\tan^2 x} dx = \int \frac{1}{\sec^2 u} \cdot \frac{1}{\sqrt{2}} \sec^2 u\, du$$

$$= \int \frac{1}{\sqrt{2}} du = \frac{1}{\sqrt{2}} u + c = \frac{1}{\sqrt{2}} \tan^{-1}(\sqrt{2}t) + c = \frac{1}{\sqrt{2}} \tan^{-1}(\sqrt{2}\tan x) + c$$

$$\therefore \int \frac{1}{1+\sin^2 x} dx = \frac{1}{\sqrt{2}} \tan^{-1}(\sqrt{2}\tan x) + c$$

2. Find $\int \frac{1}{\cos 2x - 3\sin^2 x} dx$;

$$\cos 2x - 3\sin^2 x = 1 - 2\sin^2 x - 3\sin^2 x = 1 - 5\sin^2 x$$

$$\int \frac{1}{\cos 2x-3\sin^2 x} dx = \int \frac{\sec^2 x}{\sec^2 x - 5\tan^2 x} dx;$$

$$\sec^2 x = 1 + t^2, dx = \frac{1}{1+t^2} dt$$

$$\int \frac{\text{Sec}^2 x}{\sec^2 x-5\tan^2 x} dx$$

$$= \int \frac{1}{1+t^2-5t^2} \cdot \frac{1}{1+t^2} dt$$

$$= \int \frac{1}{1-4t^2} dt;$$

but $\frac{1}{(1+2t)(1-2t)} = \frac{A}{(1+2t)} + \frac{B}{(1-2t)} = \frac{A(1-2t)+B(1+2t)}{(1+2t)(1-2t)}$

When $= \frac{1}{2}$, $1 = 0 + B(2)$, $B = \frac{1}{2}$;

$If\ t = -\frac{1}{2}, 1 = A(2), A = \frac{1}{2}$;

$$\int \frac{1}{1-4t^2} dt = \int \left\{\frac{1}{2(1+2t)} + \frac{1}{2(1-2t)}\right\} dt$$

$$= \frac{1}{2}\left(\frac{1}{2} In(1+2t) - \frac{1}{2} In(1-2t)\right) + C$$

$$= \frac{1}{4} In\left(\frac{1+2t}{1-2t}\right) + C = \frac{1}{4} In\left(\frac{1+2\tan x}{1-2\tan x}\right) + C$$

Exercise 14.

Find.

(a) $\int \frac{1}{1+\sin^2 x} dx$

(b) $\int \frac{\cos^2 x}{\cos^2 x - 4\sin^2 x} dx;$

Sub-topic 8: Splitting the numerator.

Learning Outcome

The learner should be able to integrate by splitting the numerator..

Lesson: Integration by splitting the numerator.

Splitting the numerator

When a fractional integrand with a quadratic denominator cannot be written in simple fractions, it may often be usefully expressed as two fractions by splitting the numerator.

Example 16.

Find.

1. $\int \frac{1+x}{1+x^2} dx$

$$\int \frac{1+x}{1+x^2} dx = \int \left(\frac{1}{1+x^2} + \frac{x}{1+x^2}\right) dx = \tan^{-1} x + In\sqrt{1+x^2} + C$$

2. $\int \left(\frac{5x+7}{x^2+4x+8}\right) dx$

Since $\frac{d}{dx}(x^2 + 4x + 8) = 2x + 4$

Let $5x + 7 = A(2x + 4) + B$

Equating coefficients of:

X: 5 = 2A $\quad A = \frac{5}{2}$

Constant: 7 = 4A + B $\quad , B = -3$

$$\int \left(\frac{5x+7}{x^2+4x+8}\right) dx = \int \left\{\frac{\frac{5}{2}(2x+4)}{x^2+4x+8} - \frac{3}{x^2+4x+8}\right\} dx$$

$$= \frac{5}{2} In(x^2 + 4x + 8) - 3\int \frac{1}{(x+2)^2+4} dx$$

$$= \frac{5}{2} In(x^2 + 4x + 8) - \frac{3}{2}\tan^{-1}\left(\frac{x+2}{2}\right) + C$$

This method is also appropriate for integrands of the form $\frac{a\cos x + b\sin x}{\alpha\cos x + \beta\sin x}$

Since the numerator may be expressed in the form.$A(derivative\ of\ denominator) + B(denominator)$

3. Find $\int \frac{2\cos x+3\sin x}{\cos x+\sin x}dx$;

Let $2\cos x+3\sin x \equiv A(-\sin x+\cos x)+B(\cos x+\sin x)$

When $x=0$, $2=A+B \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$

If $x=90^o, 3=-A+B \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$

$B=\frac{5}{2}$, and $A=-\frac{1}{2}$

So $\int \frac{2\cos x+3\sin x}{\cos x+\sin x}dx=\int \left\{\frac{-\frac{1}{2}(-\sin x+\cos x)}{\cos x+\sin x}+\frac{\frac{5}{2}(\cos x+\sin x)}{\cos x+\sin x}\right\}dx=-\frac{1}{2}In(\cos x+\sin x)+\frac{5}{2}x+C$

Exercise 15.

Find

a) $\int \frac{2x+3}{x^2+2x+10}dx$

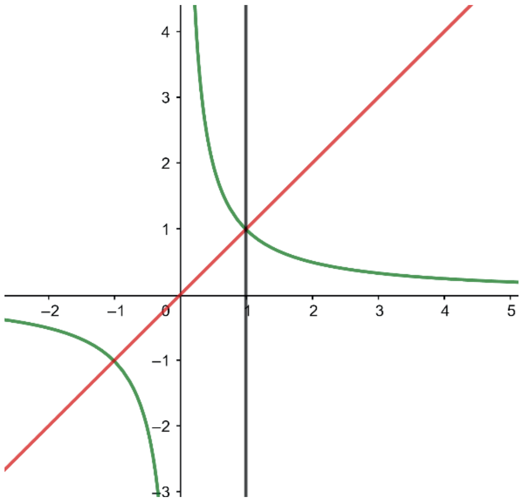
b) $\int \frac{1-2x}{\sqrt{(9-(x+2)^2)}}dx$

c) $\int \frac{\sin x}{\cos x+\sin x}dx$

d) $\int \frac{2\cos x+9\sin x}{3\cos x+\sin x}dx$

Improper integrals

There are two types of integrals to be discussed under this heading, and we shall consider them in terms of the area under a curve.



The sketch shows part of the curve $y = \frac{1}{x}$to which the $x - axis$ is an asymptote.

The area under this curve from $x = 1$ to $x = t\ (t > 1) is\ \int_0^t \frac{1}{x^2} dx = \left[-\frac{1}{x}\right]_0^t = 1 - \frac{1}{t}$

The area under this curve$y = \frac{1}{\sqrt{1-x^2}}\ from\ x = 0\ to\ x = t. (0 < t < 1) is\ \int_0^t \frac{1}{\sqrt{1-x^2}} dx = [\sin^{-1} x]_0^t =$ $\sin^{-1} t$

$$as\ t \to 1, this\ area\ \to \frac{\pi}{2}$$

$$\int_0^1 \frac{1}{\sqrt{1-x^2}} dx = [\sin^{-1} x]_0^1 = \sin^{-1} 1 = \frac{\pi}{2}$$

2. If $S \equiv \int_0^{\frac{\pi}{2}} \frac{\sin\theta}{\cos\theta + \sin\theta} d\theta\ and\ C \equiv \int_0^{\frac{\pi}{2}} \frac{\cos\theta}{\cos\theta + \sin\theta} d\theta$ prove that $S = C = \frac{\pi}{4}$

Using numerator $= A(derivative\ of\ denominator) + B(denominator)$
Let $\sin\theta \equiv A(-\sin\theta + \cos\theta) + B(\cos\theta + \sin\theta)$
$\sin\theta = (-A + B)\sin\theta + (A + B)\cos\theta; \cos\theta : A + B = 0\ and\ -A + B = 1, so\ A = -\frac{1}{2} and\ B = \frac{1}{2}$

$$\int_0^{\frac{\pi}{2}} \frac{\sin\theta}{\cos\theta + \sin\theta} d\theta = \int \left(\frac{-\frac{1}{2}(-\sin\theta + \cos\theta)}{\cos\theta + \sin\theta} + \frac{\frac{1}{2}(\cos\theta + \sin\theta)}{\cos\theta + \sin\theta} \right) d\theta$$

$$= \left[-\frac{1}{2} In(\cos\theta + \sin\theta) + \frac{1}{2}\theta \right]_0^{\frac{\pi}{2}}$$

$$= -\frac{1}{2} In\,1 + \frac{\pi}{4} — \frac{1}{2} In\,1 - 0 = \frac{\pi}{4}$$

Topic 16: Differential Equations

Sub-topic 1: Differential Equations

Learning Outcomes

The learner should be able to:

i) identify a differential equation.
ii) form a differential equation.
iii) state the order of a differential equation.
iv) find the general and particular solution of a differential equation.

Lesson: Forming and identifying differential equations (d.e)

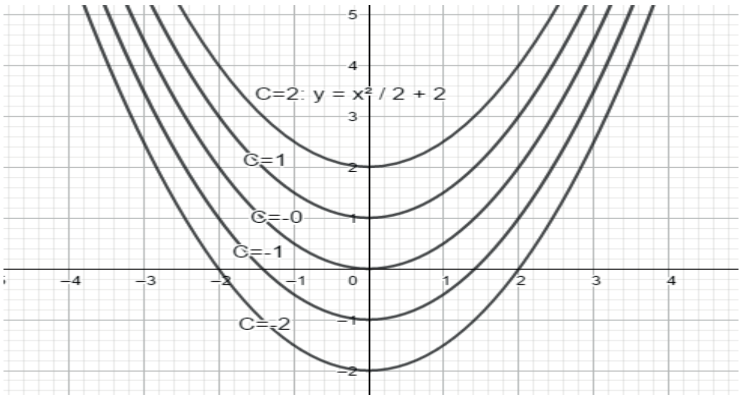
The general problem

An equation containing any differential coefficients such as $\frac{dy}{dx}, \frac{d^2y}{dx^2}$ is called a differential equation; a solution of such an equation is an equation relating x and y containing no differential coefficients. Given the differential equation $\frac{dy}{dx} = 3$, we obtain the ***general solution*** of all straight lines of $gradient$ 3. If the data also includes the fact that $y = 5$ when $x = 1$, we can determine that $c = 2$, and we obtain the **particular solution** $y = 3x + 2$.

Thus, in simple graphical terms;

a) A ***differential equation*** defines some property common to a family of curves,
b) The ***general solution***, involving one or more arbitrary constants, is the equation of any member of the family
c) A ***particular solution*** is the equation of one member of the family.

Consider the differential equation $\frac{dy}{dx} = x$, the solution of the differential equation is $y = \frac{1}{2}x^2 + c$. (In this context, the constant of integration, c, is usually called an ***arbitrary constant***)



The general solution represents all the curves with similar variable term but different constants as shown in the diagram.

Order of a DE.

The **order** of a differential equation is determined by the highest differential coefficient present.

For example $\frac{d^2y}{dx^2} = 0$ and $\frac{d^2s}{dt^2} = a$ are of the second order, whereas those $\frac{dy}{dx} = 3$and $\frac{dy}{dx} = \frac{x}{y}$are of the first order.

Since each step of integration introduces one arbitrary constant, it is in general true that the order of a differential equation gives us the number of arbitrary constants in the general solution.

This suggests that from an equation involving x, y and n arbitrary constants there may be formed (by differentiating n times and eliminating the constants) a differential equation of order n.

Example 1

1. Given that $y = Ae^{2x}$ is the solution of a differential equation,
   (a) find the differential equation.
   (b) state the order of the differential equation.

(a) $y = Ae^{2x}$................(1)

Notice that it has one arbitrary constant, so, we differentiate once and eliminate A.

$\frac{dy}{dx} = 2Ae^{2x} \Rightarrow \frac{1}{2}\frac{dy}{dx} = Ae^{2x}$...................(2)

(2) – (1)

$\frac{1}{2}\frac{dy}{dx} = Ae^{2x}$

$-(y = Ae^{2x})$

$\frac{1}{2}\frac{dy}{dx} - y = 0$

$\frac{dy}{dx} - 2y = 0$ is the DE

(b) It is a first order DE.

2. Given that $y = Acosx - Bsinx$ is the solution of a differential equation,
   (a) find the differential equation.
   (b) state the order of the differential equation.

(a) $y = Acosx - Bsinx$.................(1)

$\frac{dy}{dx} = -Asinx - Bcosx$

$\frac{d^2y}{dx^2} = -Acosx + Bsinx$.............(2)

(1) + (2)

$\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$is the differential equation.

(b) It is a second order DE.

Exercise 1

If each of the following equations is the solution of a differential equation,
   (a) find the differential equation.
   (b) state the order of the differential equation.

1. $y = Acos2x + Bsin2x$
2. $y = Ae^{3x}$

Lesson :Solving differential equations.

First order exact equations.

One side is an exact derivative(recall work on implicit differentiation.)

Example.2

Solve the differential equation $2xy\frac{dy}{dx} + y^2 = e^{2x}$

the L.H.S is $\frac{d}{dx}(xy^2)$

and the equation may be solved by integrating each side with respect to x, it is called an exact equation and the solution is $xy^2 = \frac{1}{2}e^{2x} + A$

Exercise 2

Solve the following exact equations:

a) $x^2\frac{dy}{dx} + 2xy = 1,$

b) $\frac{t^2}{x}\frac{dy}{dt} + 2t\,In\,x = 3\cos t,$

c) $x^2\cos u\frac{du}{dx} = 2x\sin u = \frac{1}{x},$

d) $e^y + xe^y\frac{dy}{dx} = 2$

Separable DE's.

The solutions of $\frac{dy}{dx} = f(x)$ and $\frac{dy}{dx} = f(y)$ which may be written $\frac{dx}{dy} = \frac{1}{f(y)}$ depend upon the integrals $\int f(x)\,dx$ and $\int \frac{1}{f(y)}dy$. There are other differential equations equally susceptible to direct integration once they have been written in a suitable form by separating the variables.

Consider $\frac{dy}{dx} = xy$. We write this as $\frac{1}{y}\frac{dy}{dx} = x$, then integrating each side with respect to x

$$\int \frac{1}{y}\frac{dy}{dx}\,dx = \int x\,dx; \quad But\ \int f(y)\,dy = \int f(y)\frac{dy}{dx}dx; \ \therefore \int \frac{1}{y}dy = \int x\,dx$$

$$\therefore In\,y + c = \frac{1}{2}x^2\,; \quad In\,y + In\,k = \frac{1}{2}x^2 \ \mid \quad In\,ky = \frac{1}{2}x^2$$

$$ky = e^{x^2/2}\ ; y = \frac{1}{k}e^{x^2/2}\,; \therefore y = A\,e^{x^2/2}$$

Example 3.

Solve $x^2\frac{dy}{dx} = y(y-1)$

Separating variables; $\int \frac{1}{y(y-1)}\,dy = \int \frac{1}{x^2}\,dx\,;$

$$\therefore \int \left\{\frac{1}{y-1} - \frac{1}{y}\right\}dy = \int \frac{1}{x^2}dx$$

$$\therefore In\frac{k(y-1)}{y} = -\frac{1}{x} \ \ or\ k(y-1) = y\,e^{-1/x}$$

Exercise 3.

Solve the following differential equations

a) $\frac{dy}{dx} = \frac{x}{y}$

b) $\frac{dy}{dx} = \frac{y}{x}$

c) $\frac{dy}{dx} = xy$

d) $x\,\frac{dy}{dx} = \tan y$

e) $e^{-x}\frac{dy}{dx} = y^2 - 1$

f) $\sqrt{(x^2+1)}\frac{dy}{dx} = \frac{x}{y}$

Example 4.

Find the particular solutions of the differential equation

$cosec\,x\frac{dy}{dx} = e^x\,cosec\,x + 3x$. Given by the conditions

(a) y=0 when x=0

(b) y=3 when $x = \frac{\pi}{2}$

(a)$cosec\,x\frac{dy}{dx} = e^x\,cosec\,x + 3x$;

$\frac{1}{\sin x}\frac{dy}{dx} = \frac{e^x}{\sin x} + 3x$;

$\frac{dy}{dx} = e^x + 3x\sin x$

$y = \int(e^x + 3x\sin x)dx$

$= e^x + 3\int x\sin x\,dx$;

$let\ u = x, \frac{du}{dx} = 1; \frac{dv}{dx} = \sin x,\ \ v = -\cos x$

$y = e^x - 3x\cos x + 3\sin x + A$

y=0 when x=0 ; $0 = 1 + A, A = -1$;

$\therefore y = e^x - 3x\cos x + 3\sin x - 1$

(b) y=3 when $x = \frac{\pi}{2}$; $3 = e^{\pi/2} - 0 + 3 + A; A = e^{\pi/2}$;

$\therefore y = e^x - 3x\cos x + 3\sin x + e^{\pi/2}$

Exercise 4

Solve the following differential equations.

(a).$(1 + \cos 2\theta\ \frac{dy}{d\theta} = 2; y = 1, when\ \theta = \frac{\pi}{4}$

(b). $\frac{dy}{dx} = x(y-2), y = 5\ when\ x = 0$

First order linear DE's.

Thus a first order linear equation is of the form.$\frac{dy}{dx} + Py = Q$

Where P, Q are functions of x or constants.

Integrating Factors

There are some differential equations which are not exact as they stand, but which may be made so by multiplying each side by an **integrating factor**.

Example 5:

Solve$xy\frac{dy}{dx} + y^2 = 3x$,;

We cannot separate the variables

$\frac{d}{dx}(xy^2) = y^2 + 2xy\frac{dy}{dx}$;

$\frac{d}{dx}(x^2y^2) = 2xy^2 + 2x^2y\frac{dy}{dx}$

$= 2x\left(y^2 + xy\frac{dy}{dx}\right) = 2x$

The required integrating factor is 2x

$multiplying\ each\ side\ by\ 2x; 2x^2y\ \frac{dy}{dx} + 2xy^2 = 6x^2$

Therefore, $x^2y^2 = 2x^3 + A$

Thus a first order linear equation is of the form.$\frac{dy}{dx} + Py = Q$

Where P, Q are functions of x or constants.

Let us assume that the general first order linear equation given above can be made into an exact equation by using the integrating factor R, a function of x. if this is so,

$R\frac{dy}{dx} + RPy = RQ$……………………………………………………………….(1) is an exact equation, and it is apparent from the first term that the L.H.S of (1) is $\frac{d}{dx}(Ry) = R\frac{dy}{dx} + y\frac{dR}{dx}$.

Thus (1) may also be written $R\frac{dy}{dx} + y\frac{dR}{dx} = RQ$…………………….(2)

Equating the second terms on the L.H.S of (1) and (2) $y\frac{dR}{dx} = RPy$; $\therefore \frac{dR}{dx} = RP$

Separating the variables$\int \frac{1}{R}\ dR = \int P\ dx$; $InR = \int P\ dx$; $R = e^{\int Pdx}$

Thus the required integrating factor is $\boldsymbol{e^{\int Pdx}}$.

The initial assumption that an integrating factor exists is therefore justified provided that it is possible to find $\int \boldsymbol{Pdx}$**.**

Example 5.

1.Solve the differential equation $\frac{dy}{dx} + 3y = e^{2x}$, given that $y = \frac{6}{5}$ when $x = 0$.

The integrating factor is $\boldsymbol{e^{\int 3dx}} = e^{3x}$**,**

multiplying each side of the given equation by $\boldsymbol{e^{3x}}$

$e^{3x}\frac{dy}{dx} + 3e^{3x}y = e^{5x}$;

$$\frac{d}{dx}(e^{3x}y) = e^{5x};$$
$$e^{3x}y = \int e^{5x}dx$$
$$; \ e^{3x}y = \frac{1}{5}e^{5x} + A$$

But y=6/5 when x=0

$\therefore \frac{6}{5} = \frac{1}{5} + A, \ \therefore A = 1;$

$Therefore\ the\ particular\ solution\ is\ y = \frac{1}{5}e^{2x} + e^{-3x}$

2, Solve $\frac{dy}{dx} + y\cot x = \cos x$

The integrating factor is $e^{\int \cot x\, dx} = e^{In \sin x} = \sin x$

Multiplying each side of the given equation by sin x,

$\text{Sin}\, x\, \frac{dy}{dx} + y\cos x = \cos x \sin x$

$\frac{d}{dx}(ysin\ x) = \cos x \sin x$ (note: LHS is always derivative of y× integrating factor)

$\int \frac{d}{dx}(ysin\ x)dx = \int \cos x \sin x\, dx$

$y\sin x = \frac{1}{2}\sin^2 x + A$

Exercise 6.

1. Solve $\frac{dy}{dx} + y + 3 = x$
2. Solve $\frac{dy}{dx} + 2xy = x$
3. Solve $\frac{dy}{dx} - y\ tan\ x = x$
4. Solve $x\frac{dy}{dx} - y = x^3e^{x^2}$

Homogeneous DE's

These are DE's where the dimensions of the terms are equal.

In the differential equation $x\frac{dy}{dx} - y = x$

$x\frac{dy}{dx}$ has dimension L , y has dimension L and x has dimension L. The dimensions are equal so the DE is first order homogeneous.

One of the variables is expressed as a product of the other and a new variable.

i.e. $y = vx \quad \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

Example 7

Solve $x\frac{dy}{dx} - y = x$

Let $y = vx \quad \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

$x\left(v + x\frac{dv}{dx}\right) - vx = x$

$x\frac{dv}{dx} = 1$ becomes separable.

$dv = \frac{1}{x}dx$

$$\int dv = \int \frac{1}{x} dx$$
$$v = lnx + c$$
$$\frac{y}{x} = lnx + c$$

Exercise 7

Solve

(a) $x\frac{dy}{dx} - y = 2x$

(b) $x^2\frac{dy}{dx} + xy = x^2 + y^2$

(c) $3x^2\frac{dy}{dx} = x^2 + y^2$ if y = 2 when x = 1.

Sub-topic 2: Application of Differential Equations

Learning Outcome

The learner should be able to form and solve differential equations related to natural occurrences.

Lesson: Application of differential equations.

It is known that radioactive substances decay, at a rate which is proportional to the amount of the radioactive substance present. If we use $\boldsymbol{x}$ to represent the amount present at time $\boldsymbol{t}$, we can express this in the form of a differential equation, namely

$\boldsymbol{\frac{dx}{dt}} = -\boldsymbol{kx}$ where k is a positive constant.

For different substances, the rate of decay is different, it is usual to quote the **half-life** of the substance, that is, the time it takes for half of the original quantity to decay.

For radium the half-life about 1600 years, we shall now solve the differential equation, that is, express $\boldsymbol{x}$ as a function of$\boldsymbol{t}$, and hence find the value of **k**. $\boldsymbol{\frac{dx}{dt}} = -\boldsymbol{kx}$

Separating the variables gives$\int \frac{1}{x} dx = \int -k\, dt$, and integrating,

$In\, x = -kt + c$; $\therefore x = e^{-kt+c} = e^c \times e^{-kt}$, and replacing e^c by A, we can write $x = A\, e^{-kt}$

This is the general solution of the differential solution (This particular differential equation is extremely common, and, unless specific instructions of the contrary are given, the solution $x = A\, e^{-kt}$.

When $t = 0, x = A$, in other words A is the original value of x, it is convenient to write this as x_0 , so $x = x_0 e^{-kt}$.

Now, we are told that when $t = 1600, x = \frac{1}{2}x_0$. Consequently,

$$\frac{1}{2}x_0 = x_0 e^{-1600k} \;;\; \therefore \frac{1}{2} = e^{-1600k} \;;\quad i.e. \quad e^{1600k} = 2,$$

Taking natural logarithms,$1600k = In\, 2$; $k = \frac{In\, 2}{1600}$

Hence the solution can be expressed

$$x = x_0 e^{\left(\frac{-In\, 2}{1600}\right)t} \;; But\; e^{In\, 2} = 2;\; hence\; x = x_0 (e^{In\, 2})^{-t/1600} = x_0 2^{\,-t/1600}$$

Finally when t=200,$x = x_0 2^{-1/8} = 0.917x_0$
In other words, after 200 years 91.7% of the original radioactive radium still exists.

Exercise 8

According to Newton's law of cooling, the rate at which the temperature of a body falls is proportional to the amount by which its temperature exceeds that of its surroundings. Suppose the temperature of an object falls from 200^0 to 100^0 in 40 minutes, in a surrounding temperature of 10^0. Prove that after t minutes, the temperature, T degrees, of the body is given by $\boldsymbol{T = 10 + 190e^{-kt}}$ where**=$^1/_{40}$In ($^{19}/_9$).** Calculate the time it takes to reach 50^0

$$Let\ \frac{dT}{dt} = -k(T-10);\ \frac{dT}{T-10} = -kdt;\ \ \int \frac{dT}{T-10} = \int -kdt\,;\ \ In(T-10) = -kt + c$$

Term 2

Topic 17: Inequalities

Learning Outcomes

The learner should be able to:

i) identify and illustrate the solution of linear inequalities on a number line.
ii) write the solutions using set notation, interval notation and inequality notation.
iii) solve simultaneous linear inequalities.
- solve quadratic inequalities.
- sketch graphs of inequalities.
- sketch modulus of inequalities.

Sub-topic: Linear and Quadratic Inequalities

Inequalities

For a quadratic, say $x^2 - 3x - 4 < 0$

Completing squares

$x^2 - 3x - 4 < 0$; $\left(x - \frac{3}{2}\right)^2 - \frac{9}{4} - 4 < 0$; $\left(x - \frac{3}{2}\right)^2 < \frac{25}{4}$; $-\frac{5}{2} < \left(x - \frac{3}{2}\right) < \frac{5}{2}$; $-1 < x < 4$

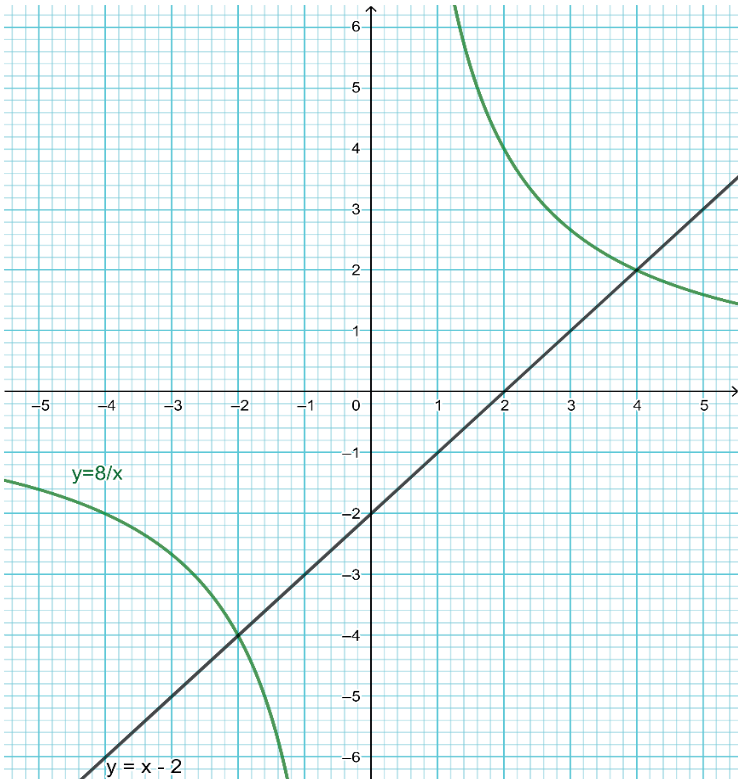
Signs of the factors; $x^2 - 3x - 4 < 0$; $(x - 4)(x + 1) < 0$

	$x < -1$	$-1 < x < 4$	$x > 4$
$x - 1$	$-$	$-$	$+$
$x + 1$	$-$	$+$	$+$
$(x - 4)(x + 1)$	$+$	$-$	$+$

$\therefore\ -1 < x < 4.$

1. Find the range (or ranges) of values that x can take if $x - 2 < \frac{8}{x}$;By sketching

In the method we sketch $y = x - 2\ and\ y = \frac{8}{x}$ on the same graph



Important points will be where these lines meet.

At points of intersection $x-2=\frac{8}{x}$ which provided $x \neq 0$, gives $x^2-2x-8=0$ giving $x=-2$ or $x=4$.

For $x-2<\frac{8}{x}$, we look for x values for which the line $y=x-2$ is lower than curve $y=\frac{8}{x}$.

By calculation:

$x-2<\frac{8}{x}$; i.e. $x-2-\frac{8}{x}<0$ or $\frac{(x-4)(x+2)}{x}<0$

Thus for $y=\frac{(x-4)(x+2)}{x}$, the critical values of x are $4, -2$ and 0.

	$x<-2$	$-2<x<0$	$0<x<4$	$x>4$
$x-4$	$-$	$-$	$-$	$+$
$x+2$	$-$	$+$	$+$	$+$
x	$-$	$-$	$+$	$+$
y	$-$	$+$	$-$	$+$

We can see that $\frac{(x-4)(x+2)}{x}<0$ for $x<-2$ and for $0<x<4$.

2. Find the solution set of the inequality $\frac{x+4}{x+1} - \frac{x-2}{x-4} < 0$ or $\frac{x-14}{(x+1)(x-4)} < 0$

	$x < -1$	$-1 < x < 4$	$4 < x < 14$	$x > 14$
$x - 14$	$-$	$-$	$-$	$+$
$x + 1$	$-$	$+$	$+$	$+$
$x - 4$	$-$	$-$	$+$	$+$
$\frac{x-14}{(x+1)(x-4)}$	$-$	$+$	$-$	$+$

Thus for $\frac{x+4}{x+1} < \frac{x-2}{x-4}$, we must have $x < -1$ or $4 < x < 14$ or in set notation

$$\{x \in \mathbb{R}: x < -1 \; or \; 4 < x < 14\}$$

3. Find the range (or ranges) of values x can take if $\frac{x-2}{x^2-x+1} > 0$

Since $x^2 - x + 1 = \left(x - \frac{1}{2}\right)^2 + \frac{3}{4}$, the denominator of $\frac{x-2}{x^2-x+1}$ is always positive. Thus the sign of $\frac{x-2}{x^2-x+1}$ depends on the sign of the numerator $x - 2$.

Now $x - 2 > 0 \; if \; x > 2$, hence $\frac{x-2}{x^2-x+1} > 0$ if $x > 2$.

Modulus inequalities:

We know that $|3x + 1| > 8$

$$3x + 1 < -8 \; or \; 3x + 1 > 8; \quad x < -3 \; or \; x > 2\tfrac{1}{2}.$$

4. Find the values of x such that $|2x - 3| > |x + 3|$. Since both sides of the inequality are positive (or zero) for all real values of x we can square both sides of the inequality.

i.e. $(2x - 3)^2 > (x + 3)^2$; $3x^2 - 18x > 0$; $3x(x - 6) > 0$

Given $x < 0 \; or \; x > 6$

	$x < 0$	$0 < x < 6$	$x > 6$
x	$-$	$+$	$+$
$x - 6$	$-$	$-$	$+$
$x(x - 6)$	$+$	$-$	$+$

Thus for $|2x - 3| > |x + 3|$ we must have $x < 0, x > 6$

Solve the inequalities

1. $(2x - 3)^2 > 1$
2. $4x^4 - 17x^2 + 4 < 0$
3. $\frac{3x+1}{2x-5} < 1$
4. $x > \frac{4-x}{x-1}$
5. $x - 2 < \frac{6x-9}{x+2}$
6. $\frac{x+1}{x-1} < \frac{x+3}{x+2}$
7. $\frac{3x}{x-8} < \frac{2x-1}{x-5}$
8. $\frac{x}{4x-8} < \frac{1}{2}$
9. $\frac{2x+3}{x+4} > \frac{x}{x-2}$
10. $|x + 3| < 4$
11. $\left|\frac{x}{x-3}\right| < 2$
12. $\left|\frac{2x-4}{x+1}\right| < 4$
13. $|2x + 5| > |x + 1|$
14. $6 - x > |3x - 2|$
15. $\left|\frac{2x+5}{x^2-4}\right| \geq \frac{1}{5}$

16. Given x is real $\&k$ is a constant show that $|k| < 3. y = \frac{3x+k}{x^2-1}$.

Topic 18: Further Curve Sketching

Sub-topic: Graphs of Functions

Learning Outcomes

The learner should be able to:

- i) find the intercepts and turning points.
- ii) define an asymptote.
- iii) state and identify different types of asymptotes.
- iv) sketch the graphs of:
  - $y = f(x)$ from curve sketching I
  - $y = \frac{1}{f(x)}$, $y = \frac{g(x)}{h(x)}$,
- v) determine regions where the curve is (not) defined and deduce the turning points.

Lesson: Further curve sketching.

1.1 Rational Functions

In this section we consider function $f(x)$ of the type $f(x) = \frac{g(x)}{h(x)}$. The five basic investigations for curve sketching used previously are still applicable for functions of this type i.e.

- i) symmetry if obvious,
- ii) intersection with axes
- iii) behaviour as $x \to \pm\infty$
- iv) $f(x)$ undefined
- v) $maximum/minimum$

Type 1 $\boldsymbol{f(x) = \frac{a}{bx+c}}$

Example

1. Make a sketch of the curve given by $y = \frac{5}{x+2}$

$\boldsymbol{x - axis}$: No value of x for which $y = 0$

$\therefore$ *no intercept with* $x - axis$.

$\boldsymbol{y - axis}$: Cuts $y - axis\ at\ \left(0,2\frac{1}{2}\right)$

$$\boldsymbol{x \to \pm\infty} : As\ x \to \pm\infty, y \to \frac{5}{x}$$

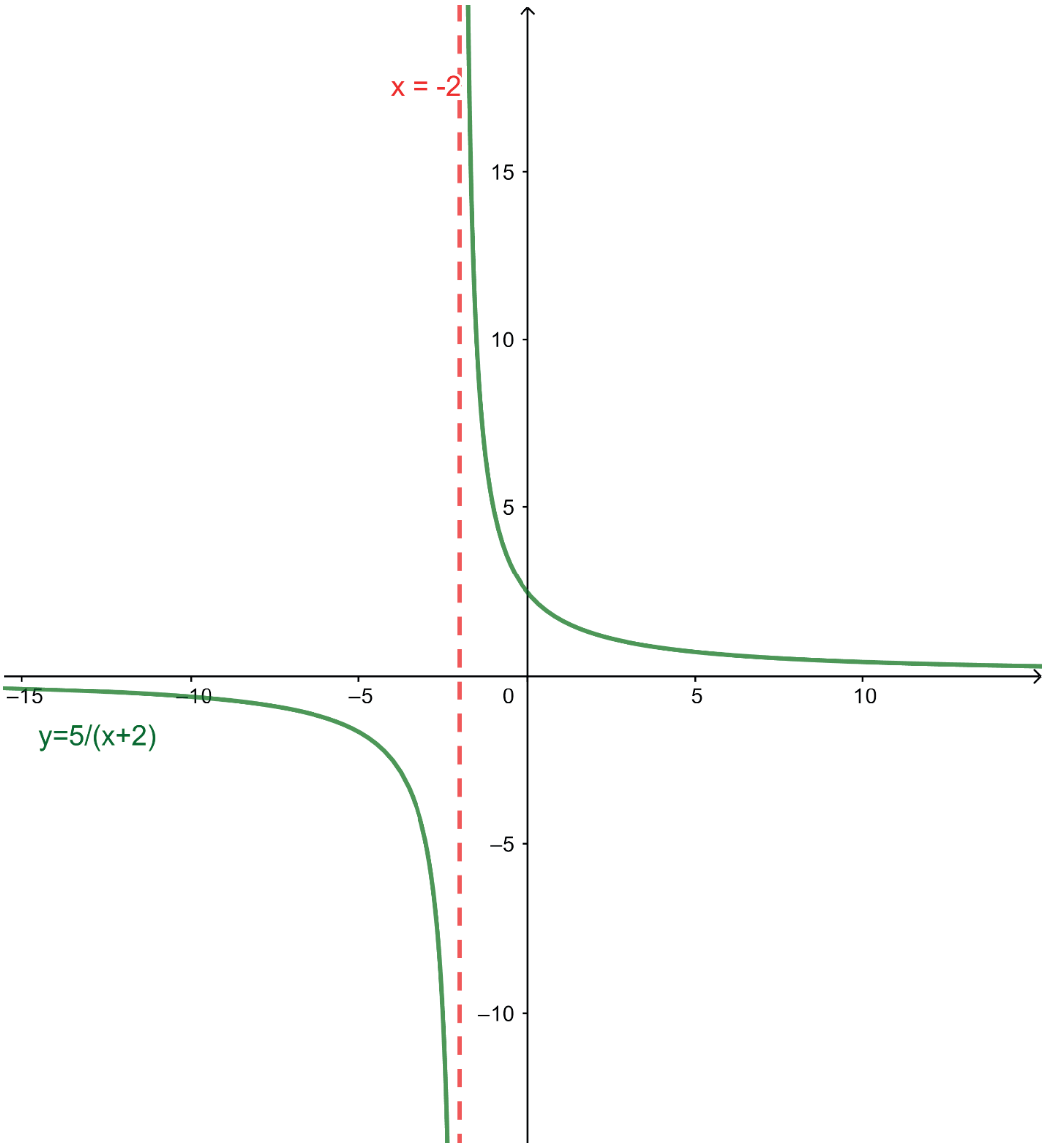
Thus for x a large positive number, y is small and positive, thus as $x \to +\infty, y \to \pm 0^+$. For x a large negative number, y is small and negative, thus as $x \to -\infty, y \to 0^-$.

$\boldsymbol{y - undefined}$, y is undefined for $x = -2$

Thus $x = -2$ is a vertical asymptote.

max/min

$y' = \frac{-5}{(x+2)^2}$, Thus no turning points and the gradient is always negative.



Type II $\boldsymbol{f(x) = \frac{ax+b}{cx+d}}$

2. make a sketch of the curve given by $y = \frac{x+3}{x-1}$

$\boldsymbol{x - axis}$: cuts $x - axis$ at $(-3,0)$

$\boldsymbol{y - axis}$: cuts $y - axis$ at $(0,-3)$

$\boldsymbol{x} \to \pm\infty$: By writing $y = \frac{1+\frac{3}{x}}{1-\frac{1}{x}}$

$As\ x \to \pm\infty, y \to 1.$

$\therefore y = 1$ is a horizontal asymptote. (It is not necessary to consider $x \to +\infty$ and $x \to -\infty$ separately for functions of this type)

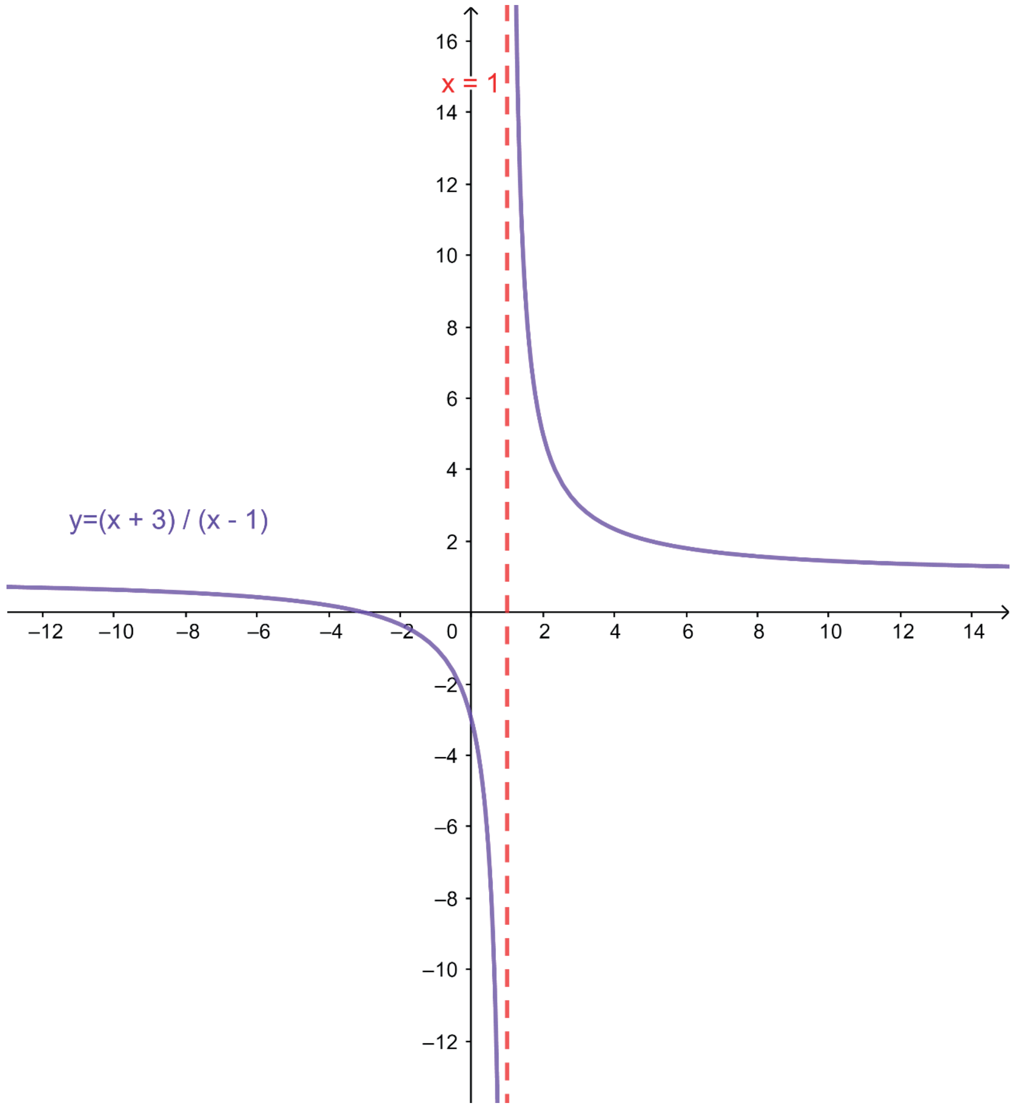
$y\ undefined$: y is undefined for $x = 1$

Thus $x = 1$ is a vertical asymptote.

$\boldsymbol{max/min.}$

$y' = \frac{1(x-1)(x+3)}{(x-1)^2} = \frac{-4}{(x-1)^2}$,

Thus no turning points and the gradient is always negative.



III $f(x) = \frac{g(x)}{h(x)}$ for $h(x)$ a quadratic function

Some functions of this type have a range that is restricted in some way for $x \in \mathbb{R}$. It is useful to examine the range of $f(x)$ first to determine whether such restrictions exist. In addition, such examinations will indicate whether any maximum or minimum points exist.

Note:

In the following examples, when $f(x)$ is an improper fraction i.e. order of $g(x) \geq order\ of\ h(x)$, then $f(x)$ is rearranged to eliminate these improper fractions when considering $x \to \pm\infty$.

3. Sketch the curve given by $y = \frac{3x+3}{x(3-x)}$

If $y = \frac{3x+3}{x(3-x)}$……………………………………………..(1)

Then $y = \frac{3x+3}{3x-x^2}$…………………………………………..(2)

Range of values of y from (2)

$$3xy - x^2y = 3x + 3$$

i.e. $x^2y + 3x(1-y) + 3 = 0$

for $y = 0$, this is a quadratic in x

thus, for real x

$$[3(1-y)]^2 - 4y(3) \geq 0$$

$$(3y-1)(y-3) \geq 0$$

We solve this inequality using the methods

	$y < \frac{1}{3}$	$\frac{1}{3} < y < 3$	$y > 3$
$3y-1$	$-$	$+$	$+$
$y-3$	$-$	$-$	$+$
$(3y-1)(y-3)$	$+$	$-$	$+$

Thus the ranges of values of y can take for real x are $y \leq \frac{1}{3}$, $y \geq 3$

We can therefore shade a region on our sketch on our sketch where the curve cannot exist. Note also that for each value of y in the allowed ranges, there will exist two distinct values of , except at $y = 3$ and $y = \frac{1}{3}$ where a repeated root will occur, and at $y = 0$ where a single root occurs. From $y \leq \frac{1}{3}$ we expect a (local) maximum at $y = \frac{1}{3}$ and $y \geq 3$ we expect a (local) minimum at $y = 3$.

$\boldsymbol{x - axis}$, cut $x - axis$ at $(-1,0)$.

$\boldsymbol{y - axis}$, No $y - axis$ intercept as y not defined for $x = 0$

$\boldsymbol{x \to \pm\infty}$, As $x \to \pm\infty, y \to \frac{3x}{-x^2}$

Thus as $x \to +\infty, y \to 0^-, asx \to -\infty, y \to 0^+$

$\boldsymbol{y\ undefined}$: $x = 0, and\ x = 3$ are vertical asymptotes

$$\boldsymbol{max/min}$$

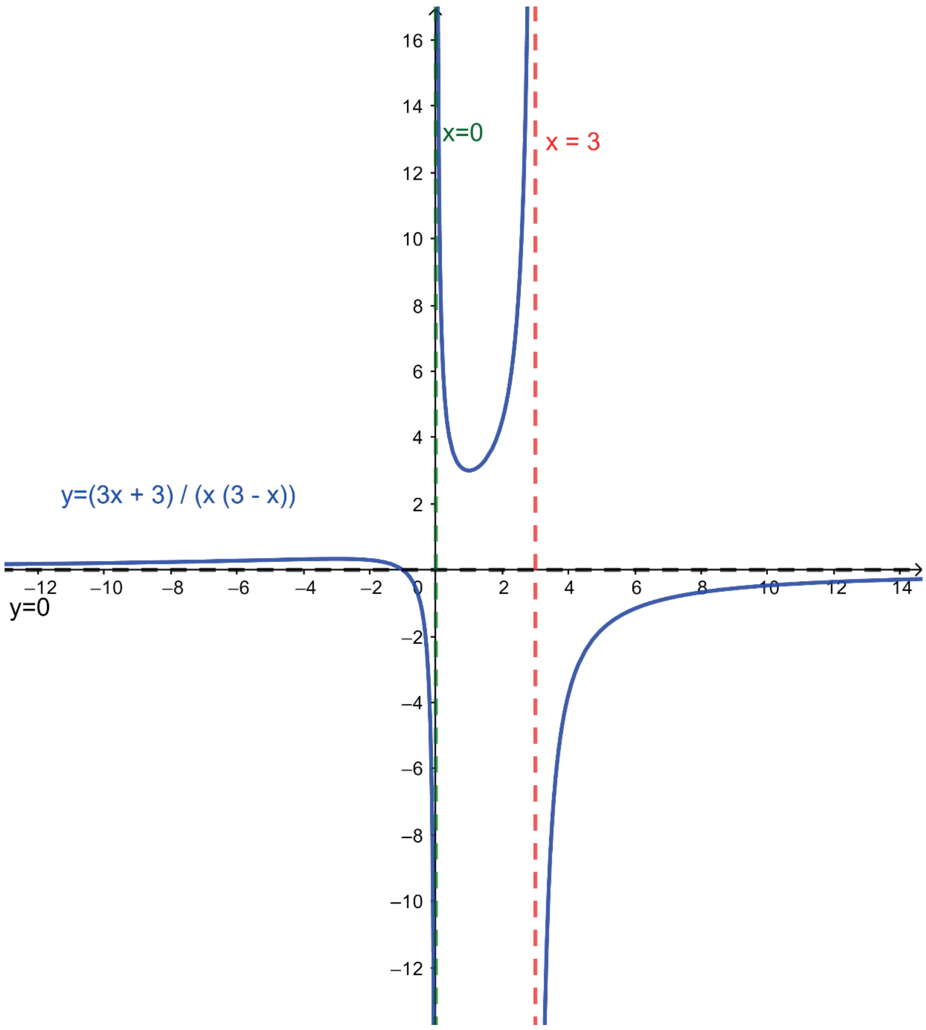
When $y = \frac{1}{3}$, $x = -3$

$$\max at \left(-3, \frac{1}{3}\right)$$

When $y = 3, x = 1$, min $(1, -3)$

For $y = 0, 3x + 3 = 0$

$$\therefore y\ can\ equal\ zero$$



Note:

For some sketches it can also be useful to determine the sign of the function throughout its domain by constructing a table.

For the last function, remembering that $f(x)$ can only change sign where the curve cuts the $x-axis$. i.e. $x=-1$, and at vertical asymptotes i.e. $(x=0, x=3)$ the table would be as shown on

	$x<-1$	$-1<x<0$	$0<x<3$	$x>3$
$3x+3$	$-$	$+$	$+$	$+$
x	$-$	$-$	$+$	$+$
$3-x$	$+$	$+$	$+$	$-$
$\frac{3x+3}{x(3-x)}$	$+$	$-$	$+$	$-$

4. Sketch the curve given by $y=\frac{(x-5)(x-1)}{(x+1)(x-3)}$. Note that R.H.S is an improper fraction.

$If\ y=\frac{(x-5)(x-1)}{(x+1)(x-3)}$..(1)

$then\ y = \frac{x^2-6x+5}{x^2-2x-3}$...(2)

$and\ y = 1 + \frac{8-4x}{x^2-2x-3}$..(3)

$$Range\ of\ values\ of\ y\ from\ (2)$$

$For\ y \neq 0, this\ is\ a\ quadratic\ in\ x.$

Thus, for real x,

$[2(3-y)]^2 - 4(y-1)(-3y-5) \geq 0.$

$$y^2 - y + 1 \geq 0$$

$$\left(y - \frac{1}{2}\right)^2 + \frac{3}{4} \geq 0$$

Which is true for all real y

Thus there is no restriction on y

For $y = 1, 4x - 8 = 0$

$$x = 2, \therefore\ y\ can\ equal\ 1$$

Note also that, for each value of y, there will exist two real distinct values of x, except where $y =$ 1 for which there is one value for x, i.e. $x = 2$.

$x - axis$: cuts $x - axis$ at $(1,0)\ and\ (5,0)$ from (1)

$y - axis$: cuts $y - axis$ at $\left(0, -1\frac{2}{3}\right)$ from 2

$\boldsymbol{x} \to \pm\infty, As\ x \to \pm\infty, y \to 1 - \frac{4x}{x^2}$ from (3)

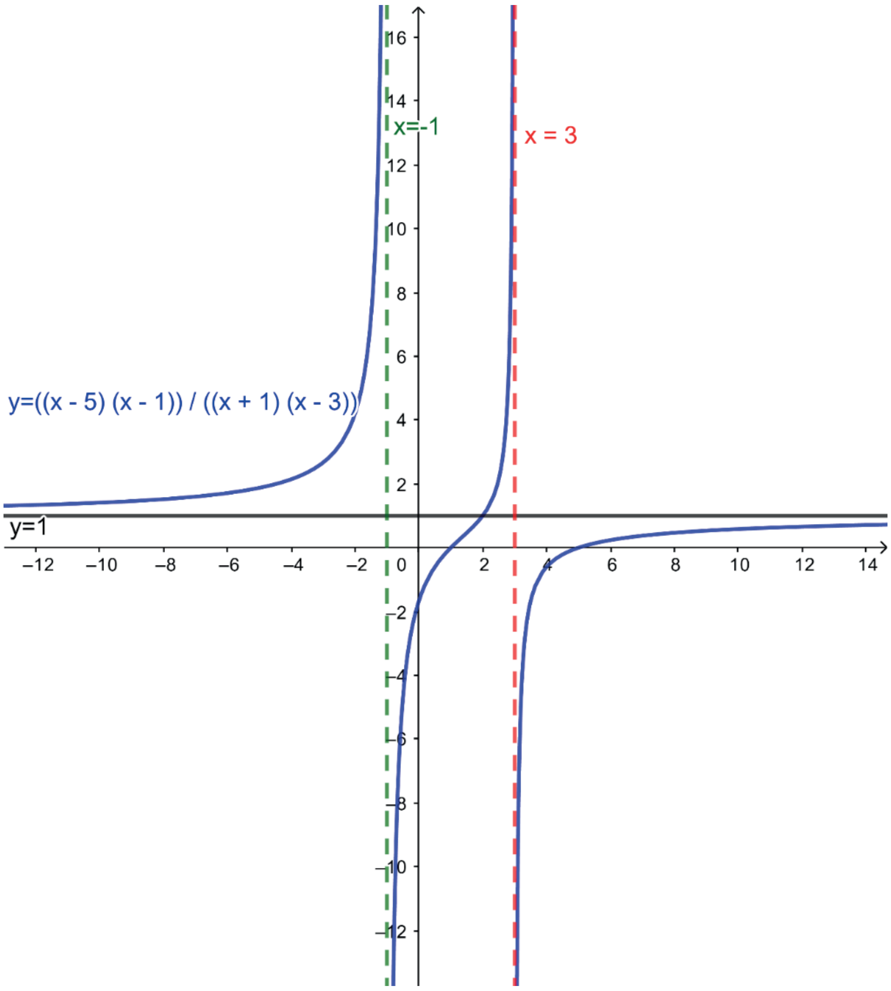
$$\therefore\ as\ x \to +\infty, y \to 1^-$$

$$as\ x \to -\infty, y \to 1^+$$

$$\boldsymbol{Undefined}:$$

$$x = -1\ and\ x = 3\ are\ vertical\ asymptotes\ from\ (1)$$

$\boldsymbol{max/min}$: Range of values suggest nomax/min.



Note: Alternatively, the horizontal asymptote could be determined by writing

$y = \frac{\left(1-\frac{6}{x}+\frac{5}{x^2}\right)}{\left(1-\frac{2}{x}-\frac{3}{x^2}\right)}$, then $as\ x \to +\infty, y \to 1$

5. Sketch the curve given by $y = \frac{12}{x^2+2x-3}$

If $y = \frac{12}{x^2+2x-3}$.. (1)

$y = \frac{12}{(x+3)(x-1)}$.. (2)

$Range\ of\ values\ of\ y, from\ 1$

$x^2y + 2xy - 12 = 0.$

$for\ y \neq 0, this\ is\ a\ quadratic\ in\ x.$

$Thus, for\ real\ x,\quad 4y^2 - 4y(-3y - 12) \geq 0$

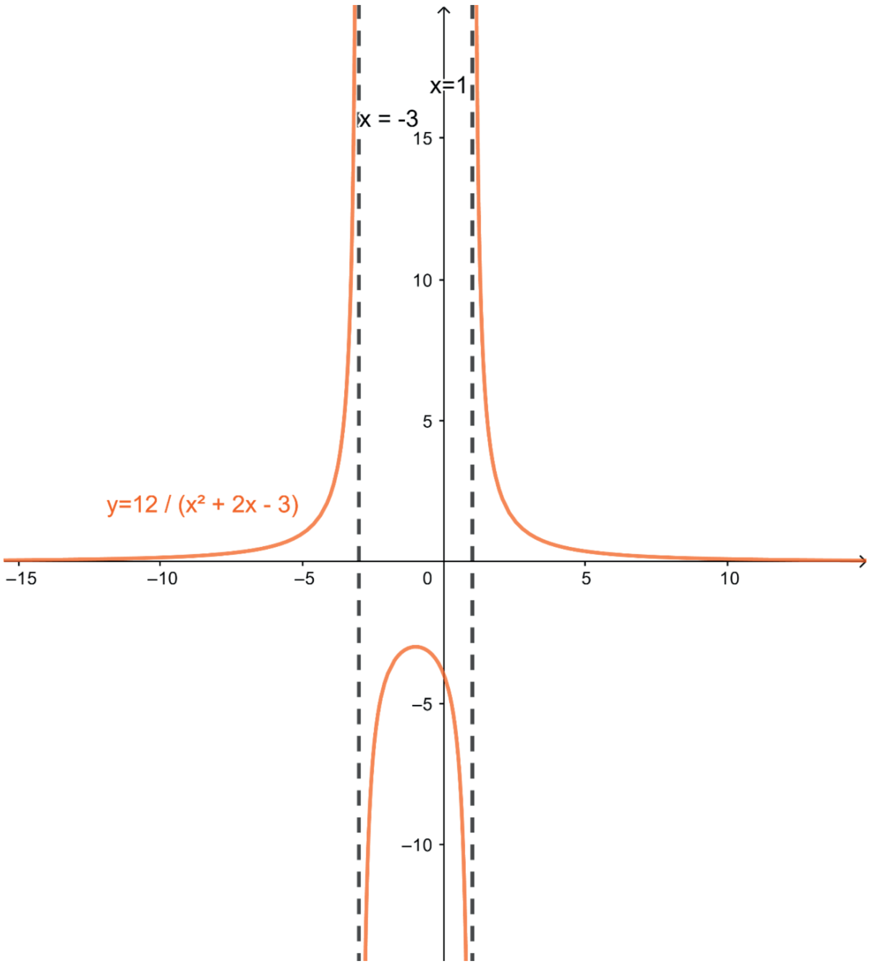
$y(y + 3) \geq 0$

	$y < -3$	$-3 < y < 0$	$y > 0$
y	$-$	$-$	$+$
$y + 3$	$-$	$+$	$+$
$y(y + 3)$	$+$	$-$	$+$

$\therefore$ The ranges of values y can take for real x are $y \leq -3, y > 0$.
$\Rightarrow y = -3$ will be a local maximum and for the remainder of the range there will be two distinct real values of x for each value of y. (for $y = 0$, we obtain $-12 = 0$. Which is impossible$y \neq 0$)
$\boldsymbol{x - axis}$: No $x - axis$ intercept.
$\boldsymbol{y - axis}$: Cuts $y - axis\ at\ (0, -4)$
$\boldsymbol{x \to \pm\infty}$: As $x \to \pm\infty, y \to \frac{12}{x^2}\ from\ (1)$

$$\therefore as\ x \to +\infty, y \to 0^+$$
$$as\ x \to -\infty, y \to 0^+$$
$$\boldsymbol{y\ undefined} :$$
$$x = -3\ and\ x = 1\ are\ vertical\ asymptotes\ from\ (2)$$
$$\boldsymbol{max/min}$$

When $y = -3, x = -1$, max $at\ (-1, -3)$



6. Sketch the curve given by $y = \frac{2-3x}{x^2+3x+3}$.
In this case the denominator does not factorize Range of values of y for $y = \frac{2-3x}{x^2+3x+3}$
We have $x^2y + 3x(y + 1) + 3y - 2 = 0$
For $y \neq 0$, this is a quadratic in x.

Thus, for real , $[3(y+1)]^2 - 4y(3y-2) \geq 0$. i.e. $(3y+1)(y-9) \leq 0$

	$y < -\frac{1}{3}$	$-\frac{1}{3} < y < 9$	$y > 9$
$3y+1$	$-$	$+$	$+$
$y-9$	$-$	$-$	$+$
$(3y+1)(y-9)$	$+$	$-$	$+$

i.e. $-\frac{1}{3} < y < 9$

$\therefore$ the range of values y can take real x is $-\frac{1}{3} \leq y \leq 9$

$y = -\frac{1}{3}$ will be a minimum and $y = 9$ a maximum.

For every other value of y in the permitted range there will correspond two distinct values of x except $y = 0$ for which there is one value of x.

For $y = 0, 3x = 2, x = \frac{2}{3}, y\ can\ equal\ zero.$

$x - axis$: cuts $x - axis$ at $\left(\frac{2}{3}, 0\right)$

$y - axis$: cuts $y - axis\ at\ \left(0, \frac{2}{3}\right)$

$$x \to \pm\infty : as\ x \to \pm\infty, y \to -\frac{3x}{x^2}$$

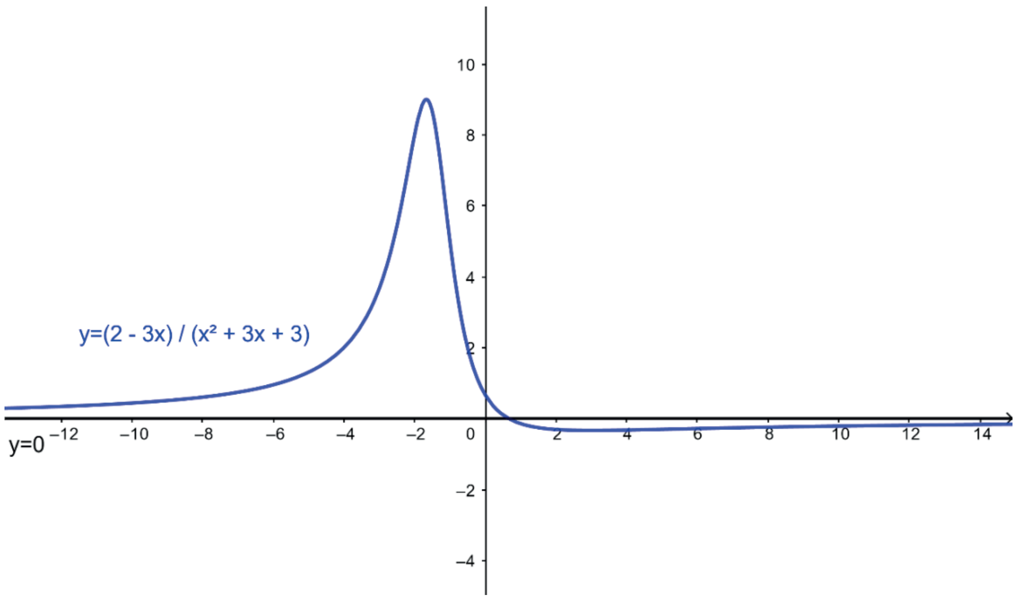
$$\therefore as\ x \to +\infty, y \to 0^-$$

$$as\ x \to -\infty, y \to 0^+$$

$y - undefined$: No values for which $x^2 + 3x + 3 = 0$

max/ min : When $y = -\frac{1}{3}, x = 3$, min $at \left(3, -\frac{1}{3}\right)$

When $y = 9, x = -1\frac{2}{3}$, max $at \left(-1\frac{2}{3}, 9\right)$



7. Example 7

Sketch the curve given by $y = \frac{x^2+4x+3}{x+2}$

Note that in this case the RHS is an improper fraction

If $y = \frac{x^2+4x+3}{x+2}$..(1)

Then $y = \frac{(x+3)(x+1)}{x+2}$... (2)

And $y = x + 2 - \frac{1}{x+2}$...(3)

Range of values of y

From (1) $x^2 + x(4-y) + 3 - 2y = 0$

Thus, for real x; $(4-y)^2 - 4(1)(3-2y) \geq 0$

$$y + 4 \geq 0$$

Which is true for all y.

Thus there is no restriction on y, for each value of y, there exists two distinct values for x.

$\boldsymbol{x-axis}$: cuts $x - axis\ at\ (-3,0)\ and\ (-1,0)$

$\boldsymbol{y-axis}$: cuts $y - axis\ at\ \left(0, 1\frac{1}{2}\right)$

$$\boldsymbol{x \to \pm\infty} : as\ x \to \pm\infty, y \to x + 2 - \frac{1}{x}$$

$$\therefore\ as\ x \to +\infty, y \to (x+2)^-$$

$$as\ x \to -\infty, y \to (x+2)^+$$

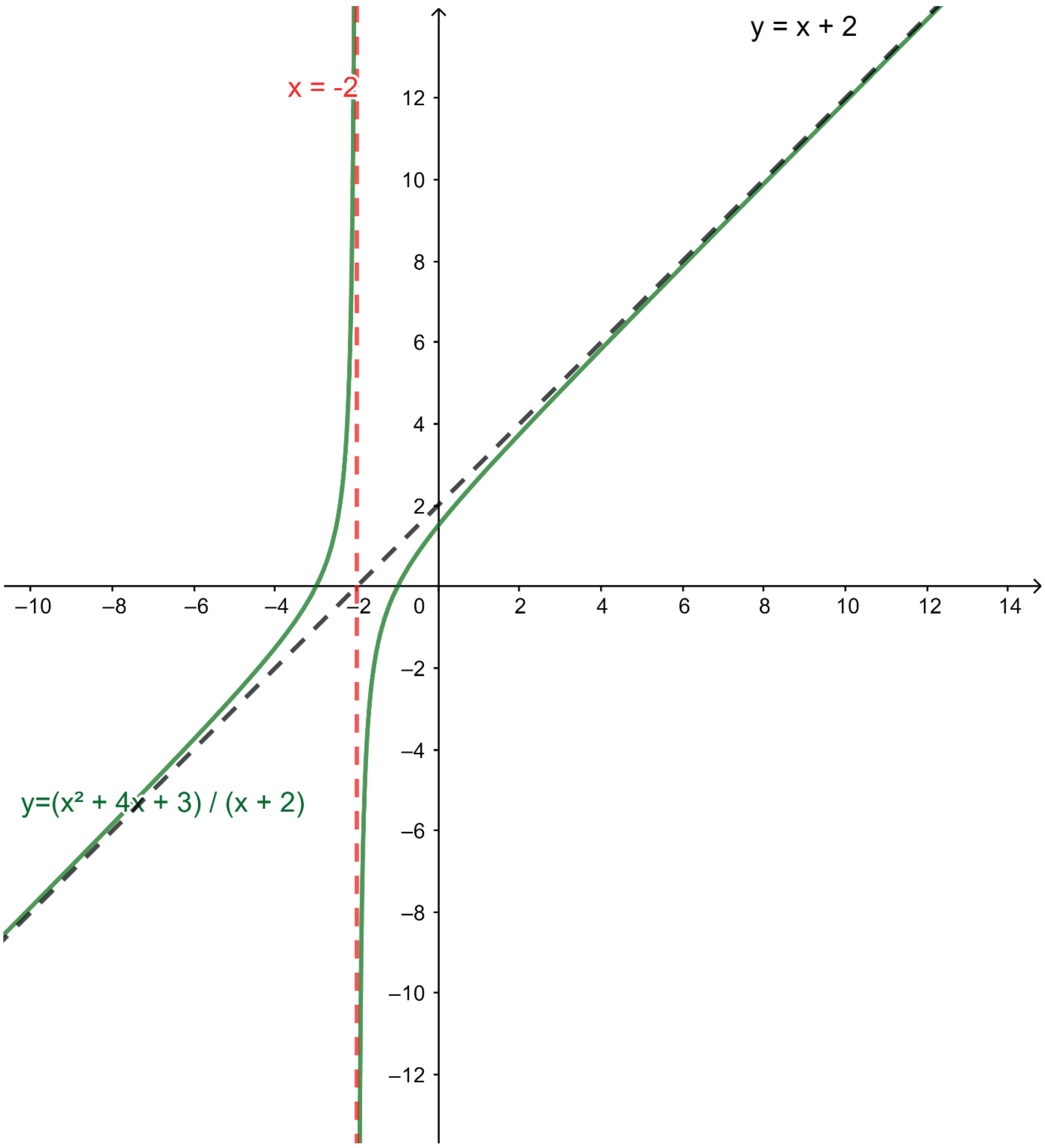
Hence $y = x + 2$ is an oblique asymptote.

$\boldsymbol{y - undefined}$: $x = -2$ as a vertical asymptote.

$$\boldsymbol{max/\ \mathrm{min}} :$$

Range of values suggest no max/min

$y' = 1 + \frac{1}{(x+2)^2}$ So gradient is always positive.


y = x + 2
x = -2
y=(x² + 4x + 3) / (x + 2)
–10
–8
–6
–4
–2
0
2
4
6
8
10
12
14
12
10
8
6
4
2
–2
–4
–6
–8
–10
–12

Topic: 19: Coordinate Geometry II

Learning Outcomes

The learner should be able to:

i) identify different types of loci.
ii) find the equation of a locus of a variable point.

Sub-topic 1: Locus

Learning Outcomes: To identify and find equation of locus.

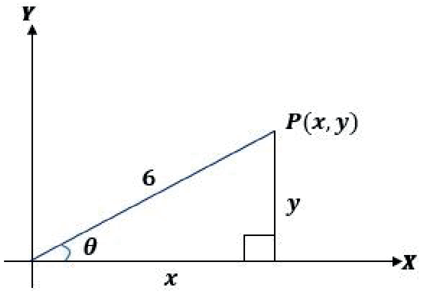
Lesson: Loci

By definition the path traced out by the point **P** as it moves according to the given conditions is called the ***locus of P*** or when possible positions of point **P** are restricted to a straight line or curve the set of such points is called a locus of **P**.

We usually use a point $P(x, y)$ to represent a moving point; any fixed point may be represented by any letter, the origin with letter O. If P is the point(x, y), the relationship between x and y which applies only to the sets of point P is called the Cartesian equation of a line.

The equation of a locus

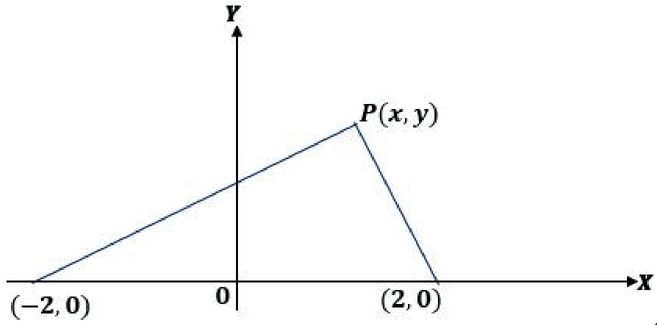
a) In the first of the introductory problem above, the given condition is$OP = 6$, so if O is the origin the equation of the locus can be obtained by applying Pythagoras theorem.



The equation of the locus is $x^2 + y^2 = 36$.

b) In the second problem, we shall take the two fixed points to be $(-2,0)$ and $(2,0)$ respectively.

Applying the usual formula for the distance between two points we obtain

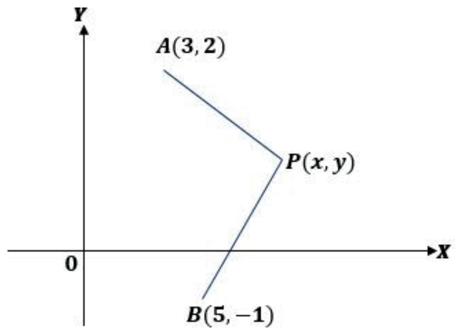


$$AP = \sqrt{(x+2)^2 + y^2} \text{ and}$$
$$BP = \sqrt{(x-2)^2 + y^2}$$

The condition which governs the movement of the point P is $\boldsymbol{AP + PB = 6}$, so the equation of the locus is

$$\sqrt{(x+2)^2 + y^2} + \sqrt{(x-2)^2 + y^2} = 6$$

Example



1. Find the equation of the locus of a point P which moves so that it is equidistant from two fixed points A and B whose coordinates are $(3,2)$ and $(5,-1)$ respectively.

Solution

Let P be the point (x, y)

Expressed geometrically, the condition satisfied by P is $\boldsymbol{AP = PB}$

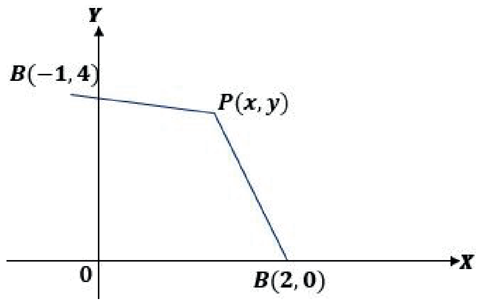
So,

$$AP = \sqrt{(x-3)^2 + (y-2)^2} \text{ and } PB = \sqrt{(x-5)^2 + (y+1)^2}$$
$$AP = PB$$
$$\text{then } AP^2 = PB^2$$
$$\Rightarrow (x-3)^2 + (y-2)^2 = (x-5)^2 + (y+1)^2$$
$$\Rightarrow x^2 - 6x + 9 + y^2 - 4y + 4 = x^2 - 10x + 25 + y^2 + 2y + 1$$
$$\Rightarrow 4x - 6y - 13 = 0$$

Therefore the equation of the locus of points equidistant from $(3,2)$ *and* $(5,-1)$ *is* $4x - 6y - 13 = 0$.

2. Find the locus of a point $\boldsymbol{P}$, whose distance from the point $(2,0)$ is twice distance from $(-1,4)$.

$$\overline{AP} = 2\overline{PB}\overline{AP}^2 = 4\overline{PB}^2$$



So

$$\overline{AP}^2 = (x-2)^2 + y^2 \ \ and \ \ \overline{PB}^2 = (x+1)^2 + (y-4)^2$$

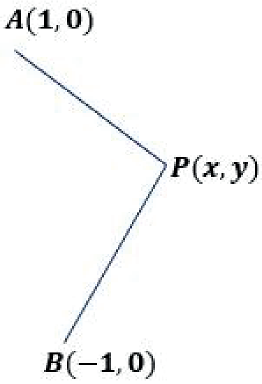
Then

$\overline{AP}^2 = 4\overline{PB}^2$ becomes

$$(x-2)^2 + y^2 = 4[(x+1)^2 + (y-4)^2]$$
$$\Rightarrow x^2 + 4x + 4 + y^2 = 4[x^2 + 2x + 1 + y^2 - 8y + 16] \Rightarrow x^2 + 4x + 4 + y^2$$
$$= 4x^2 + 8x + 4 + 4y^2 - 32y + 64$$

$\therefore 3x^2 + 3y^2 + 12x - 32y + 64 = 0$; Equation of the circle.

3. A is the point (1,0) and B is the point (−1,0). Find the locus of a point **P** which moves so that $\boldsymbol{PA + PB = 4}$.



Solution

Let the point be $P(x, y)$

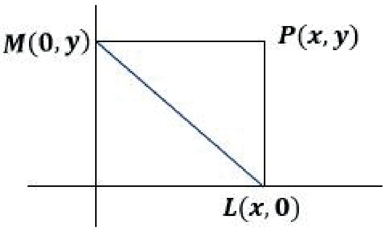
$$\overline{AP} = \sqrt{(x-1)^2 + (y)^2}$$
$$\overline{PB} = \sqrt{(x+1)^2 + (y)^2}$$

From $\boldsymbol{PA + PB = 4}$

$$\sqrt{(x-1)^2 + y^2} + \sqrt{(x+1)^2 + y^2} = 4$$
$$\Rightarrow \sqrt{(x-1)^2 + y^2} = 4 - \sqrt{(x+1)^2 + y^2}$$
$$squaring\ both\ sides$$
$$\Rightarrow (x-1)^2 + y^2 = 4^2 - 8\left(\sqrt{(x+1)^2 + y^2}\right) + (x+1)^2 + y^2$$
$$\Rightarrow x^2 - 2x + 1 + y^2 = 16 - 8\left(\sqrt{(x+1)^2 + y^2}\right) + x^2 + 2x + 1 + y^2$$
$$\Rightarrow -2x + 1 = 17 + 2x - 8\left(\sqrt{(x+1)^2 + y^2}\right)$$
$$\Rightarrow 8\left(\sqrt{(x+1)^2 + y^2}\right) = 17 + 2x - 1 + 2x$$

$$\Rightarrow 8\left(\sqrt{(x+1)^2+y^2}\right) = 16+4x$$
$$\Rightarrow \left(\sqrt{(x+1)^2+y^2}\right) = \frac{16+4x}{8} = 2+\frac{1}{2}x$$
squaring both sides
$$\Rightarrow (x+1)^2+y^2 = (2+\frac{1}{2}x)^2 = 4+2x+\frac{1}{4}x^2$$
$$\Rightarrow x^2+2x+1+y^2 = 4+2x+\frac{1}{4}x^2$$
$$\Rightarrow x^2-\frac{1}{4}x^2-2x-2x+1-4+y^2 = 0$$
$$\Rightarrow \frac{3}{4}x^2+y^2-3 = 0$$
$$\Rightarrow 3x^2+4y^2-12 = 0$$
$$\therefore 3x^2+4y^2-12 = 0$$

4. $\boldsymbol{L}$ and $\boldsymbol{M}$ are the feet of the ⊥ (perpendicular) from $\boldsymbol{P}$ on the axes. Find the locus of $\boldsymbol{P}$when it moves so that LM is of length 4 units.

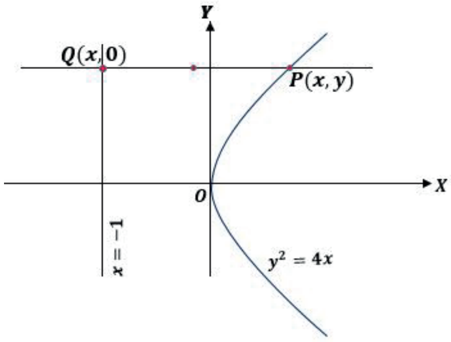


$$y^2+x^2 = 16$$

Further examples

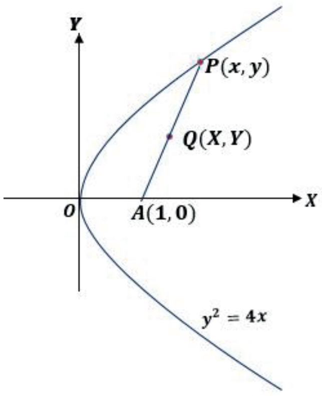
1. A line parallel to the $x-axis$ cuts the curve $y^2=4x$ at $\boldsymbol{P}$ and the line $x=-1$ at$\boldsymbol{Q}$. Find the locus of the mid-point of $\boldsymbol{PQ}$.

   Solution

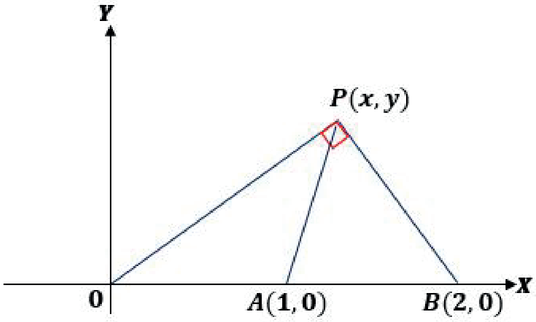


2. A variable point $\boldsymbol{P}$ moves on the curve $y^2=4x$ and A is the point(1,0). Find the locus of the mid-point of $\boldsymbol{AP}$.

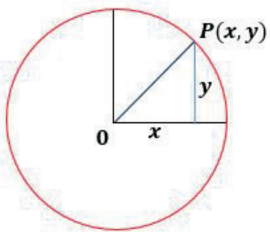
   Solution



3. ***A*** is a point $(\mathbf{1}, \mathbf{0})$ and ***B*** is the point (2,0) and ***O*** is the origin. A point ***P*** moves so that angle **BPO** is a right angle and ***Q*** is the mid-point of ***AP***. What is the locus of ***Q***?
   Solution



4. Find the equation of a circle with centre at the origin and radius 5 units.
   Solution



Tangents and normals

If a tangent touches a curve at the point ***P***, the line through ***P*** perpendicular to the tangent is called a ***normal***.

If the gradient of the tangent is $\boldsymbol{m}$, the gradient of the normal is $^{-1}/_{m}$.

Examples

1. Find the equations of the tangent and normal to the curve $y = 3x^2 - 8x + 5$, at the point where $x = 2$.
   Solution
   The gradient of the tangent, $\frac{dy}{dx}$; $y = 3x^2 - 8x + 5$; $\frac{dy}{dx} = 6x - 8$

At the point of contact $x = 2$, so $\frac{dy}{dx} = 6 \times 2 - 8 = 4$

At $x = 2, y = 3(2)^2 - 8 \times 2 + 5 = 1$

Therefore the coordinates of the point of contact are $(\mathbf{2}, \mathbf{1})$

Using $y - y_1 = m(x - x_1)$

$$\Rightarrow y - 1 = 4(x - 2)$$
$$\Rightarrow 4x - y - 7 = 0$$

Therefore the equation of the tangent is $4x - y - 7 = 0$

The normal is perpendicular to the tangent, and so, its gradient is $-\frac{1}{4}$

So the equation, $y - 1 = -\frac{1}{4}(x - 2); \Rightarrow x + 4y - 6 = 0$

2. Find the equation of tangent to the curve $y = x^3 - 3x^2$ which are parallel to the $y = 9x + 10$.

   Solution

$$\frac{dy}{dx} = 9\ from\ y = 9x + 10.$$

   Then also, from $y = x^3 - 3x^2$; $\frac{dy}{dx} = 3x^2 - 6x$; $\Rightarrow 3x^2 - 6x = 9$; $\Rightarrow x^2 - 2x - 3 = 0$; $\Rightarrow$ $(x - 3)(x + 1) = 0$

   $\Rightarrow x = 3\ or\ x = -1\ ;\ when\ x = 3\ then\ y = 0\ ;\ when\ x = -1\ then\ y = -4$

   Then using $y - y_1 = m(x - x_1)$; For $(3,0)$; $y - 0 = 9(x - 3); \Rightarrow \boldsymbol{y = 9(x - 3)}$

   For $(-1, -4)$; $y -^{-}4 = 9(x -^{-}1); \Rightarrow y + 4 = 9x - 9; \Rightarrow \boldsymbol{y = 9x - 5}$

3. Show that if the line $y = x + c$ is a tangent to the curve $x^2 + y^2 = 4$

   (Note:

   If the line $\boldsymbol{y = mx + c}$ is a tangent then the point of contact must be given by an equation with repeated roots. If the equation $\boldsymbol{ax^2 + bx + c = 0}$ has equal roots the $b^2 = 4ac$ and real roots $b^2 - 4ac \geq 0$**)**

   Solution:

$$y = x + c$$
$$x^2 + y^2 = 4$$

   So
$$(x + c)^2 + x^2 = 4$$
$$x^2 + 2cx + c^2 + x^2 = 4$$
$$2x^2 + 2cx + (c^2 - 4) = 0$$

   For equal roots

$$b^2 = 4ac$$
$$4c^2 = 4 \times 2 \times (c^2 - 4)$$
$$c^2 = 2c^2 - 8$$
$$c^2 = 8$$

4. Find the equations of the tangents to the curve $xy = 6$ which are parallel to the line $2y + 3x = 0$.

   Solution:

   The gradient of the line $2y + 3x = 0$

$$y = \frac{-3}{2}x \; so \; m = {}^{-3}/_{2}$$

Gradient of $xy = 6$

$$\Rightarrow y + x\frac{dy}{dx} = 0$$

$$\Rightarrow x\frac{dy}{dx} = -y$$

$$\Rightarrow \frac{dy}{dx} = \frac{-y}{x} = \frac{-6}{x^2}$$

$$if \; \frac{dy}{dx} = \frac{-3}{2}$$

$$\Rightarrow \frac{-6}{x^2} = \frac{-3}{2}$$

$$\Rightarrow 3x^2 = 12 \; so \; x^2 = 4, x = \pm 2$$

$$when \; x = 2, \qquad y = \frac{6}{2} = 3; \qquad \therefore Point(2,3)$$

$$when \; x = -2, \qquad y = \frac{-6}{2} = -3; \qquad \therefore Point(-2,-3)$$

∴ Equations of the tangents in the form
$y - y_1 = m(x - x_1)$;

$$y - 3 = \frac{-3}{2}(x - 2) \; and \; y + 3 = \frac{-3}{2}(x + 2)$$

$$\therefore tangents \; are \; 3x + 2y - 12 = 0 \; and \; 3x + 2y + 12 = 0$$

5. Show that if the line $y = mx + c$is a tangent to the curve $4x^2 + 3y^2 = 12$, then $c^2 = 3m^2 + 4$. (If the line $y = mx + c$ is a tangent, then the point of contact must be given by an equation with a repeated root.)

Sub-topic: 2: The Circle

Learning Outcomes

The learner should be able to:

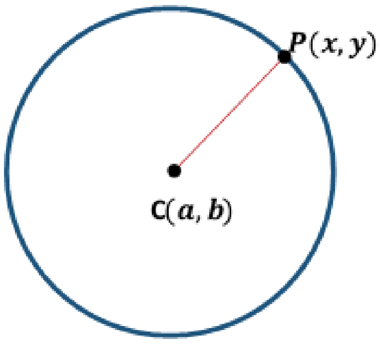
i) form and identify the equation of a circle.
ii) find the Centre and radius of a circle.
iii) find the equation of a circle given any points.
iv) determine the equation of the tangent at a given point.
v) determine the points of intersection of two circles.
vi) find the condition for external, internal and orthogonal intersection of two circles.
vii) determine the length of the tangent to a circle.

Lesson: Circle

The equation of a circle

A circle is the locus of a point which moves in a place in such a way that its distance from the fixed point (the centre) is always constant (radius)

Consider a circle with centre(a, b) with radius r and let $P(x, y)$ be any point on the circle.



$$CP = r$$
$$\overline{CP}^2 = r^2$$
$$(x - a)^2 + (y - b)^2 = r^2$$

This is the equation of the Centre(a, b) and radius r.

If Centre is (0,0)

Then,

$$x^2 + y^2 = r^2$$

a) Find the equation of the curve Centre$(-3,1)$ and radius 5

$$(x + 3)^2 + (y - 1)^2 = 5^2$$
$$x^2 + 6x + 9 + y^2 - 2y + 1 = 25$$
$$x^2 + y^2 + 6x - 2y = 15$$

b) Find the radius and the centre of the circle.

$$2x^2 + 2y^2 - 8x - 12y - 9 = 0$$
$$x^2 + y^2 - 4x - 6y - {}^9/_2 = 0$$
$$x^2 - 4x + y^2 - 6y = {}^9/_2$$

Then completing squares

$$(x - 2)^2 - 2^2 + (y - 3)^2 - 3^2 = {}^9/_2$$
$$(x - 2)^2 + (y - 3)^2 = 3^2 + 2^2 + {}^9/_2 = \frac{35}{2}$$
$$\therefore Centre\ (2,3) and\ Radius, r = \sqrt{\left({}^{35}/_2\right)}$$

Note:

The equation of a circle has the following properties

i. It is a second degree equation in $x\ and\ y$.

ii. It contains no term of xy.

iii. The coefficient of $x^2\ and\ y^2$ are equal.

In general the equation of a circle is written as $x^2 + y^2 + 2gx + 2fy + c = 0 centre\ (a, b)$

$$(x - a)^2 + (y - b)^2 = r^2$$
$$x^2 + y^2 - 2ax - 2by + a^2 + b^2 - r^2 = 0$$

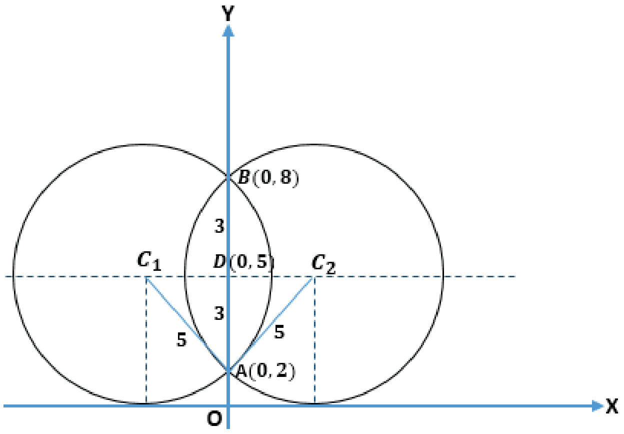
Example

Find the equation of the circles which pass through the point $A(0,2)$ *and* $B(0,8)$ which touch the $x - axis$

The centre of the circle must lie on the perpendicular bisector of the chord AB. i.e. on the line $y = 5$. Now the circle touches the $x - axis, \therefore r = 5$.

If D is the point $(0,5)$ and C is the centre of either circle,

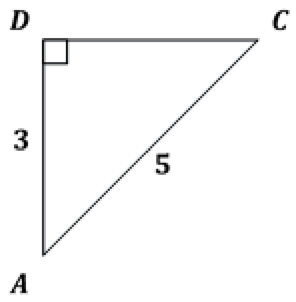
The triangle ADC



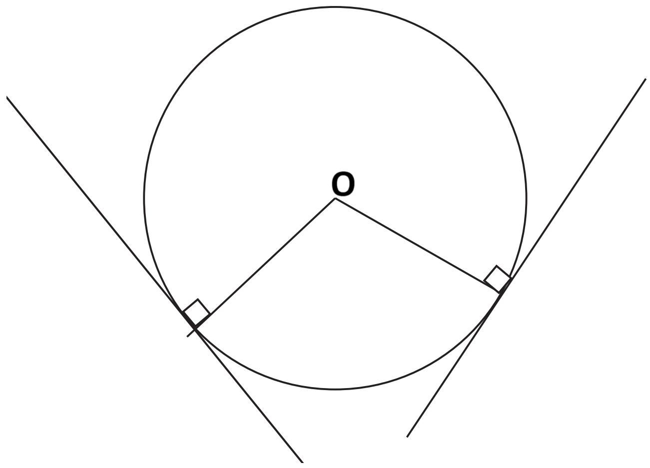
$$DC = \sqrt{5^2 - 3^2} = \sqrt{16} = 4$$

Therefore the centres of the circles are $(-4,5)$ and $(4,5)$ and so their equations are

$$(x \pm 4)^2 + (y - 5)^2 = 5^2$$
$$x^2 + y^2 \pm 8x - 10y + 16 = 0$$
$$so\ x^2 + y^2 - 8x - 10y + 16 = 0$$
$$x^2 + y^2 - 8x - 10y + 16 = 0$$



Tangents to a circle



- Any line is **tangent** to the circle if the **distance** from the centre to the line is equal to the **radius** at the point of contact.
- The radius is **perpendicular** to the equation of the tangent.

Example:

Verify that the point (3,2) lies on the circle $x^2 + y^2 - 8x + 2y + 7 = 0$, and find the equation of the tangent at this point.

Solution

Substituting the coordinates (3,2) into the equation $x^2 + y^2 - 8x + 2y + 7 = 0$.

$$L.H.S = 9 + 4 - 24 + 4 + 7 = 0 = R.H.S$$

Therefore (3,2) lies on the circle.

(the gradient of the tangent can be found from the the gradient of the radius through (3,2) and in order to find this, we obtain the coordinate of the centre of the circle.

The equation of the circle

$$x^2 - 8x + y^2 + 2y = -7$$

$$x^2 - 8x + 4^2 + y^2 + 2y + 1^2 = -7 + 4^2 + 1^2$$

$$(x - 4)^2 + (y + 1)^2 = 10$$

Therefore the centre of the circle is $(4, -1)$ hence the gradient through (3,2) is

$$\frac{-1 - 2}{4 - 3} = -3$$

Therefore the gradient of the tangent is $\frac{1}{3}$

Then using $y - y_1 = m(x - x_1)$

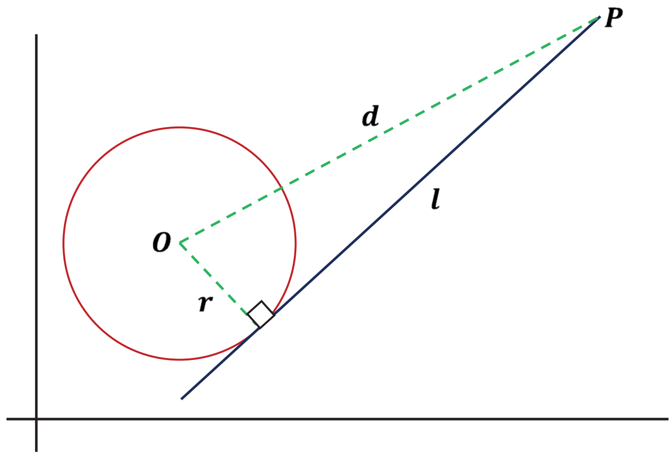
The equation of the tangent at (3,2) is

$$y - 2 = \frac{1}{3}(x - 3)$$

$$3y - 6 = x - 3$$

$$\therefore \textit{The tangent to the circle at } (3,2) \textit{is } x - 3y + 3 = 0$$

The length of a tangent from a given point



By pythagoras theorem

$$d^2 = l^2 + r^2$$

Figure

1. Find the length of the tangents to the circle $x^2 + y^2 - 4x - 8y + 5 = 0$ from (8,2)

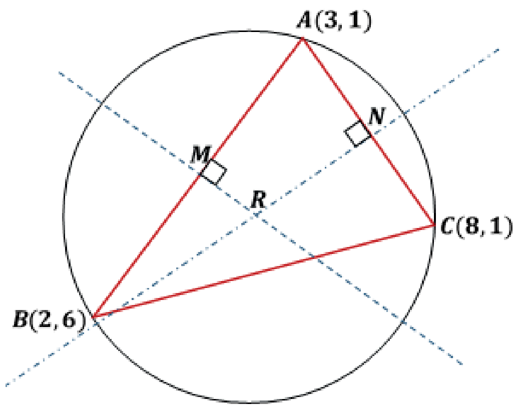
Solution: $(x - 2)^2 + (y - 4)^2 = -5 + 16 + 4$

$$(x-2)^2 + (y-4)^2 = 15$$
$$Centre\ (2,4), \qquad r = \sqrt{15}$$
$$d \text{ is the distance between the centre } (2,4) \text{ and point } (8,2)$$
$$d^2 = (8-2)^2 + (2-4)^2 = 36 + 4 = 40$$
$$l^2 = 40 - 15$$
$$l^2 = 25$$
$$l = 5$$

---

2. Find the equation of a circle which passes through three points $(3,1), (8,2)$ *and* $(2,6)$.

Solution:



$$Gradient\ AB = \frac{1-6}{3-6} = -5$$
$$Gradient\ \perp AB = \frac{1}{5}$$
$$Gradient\ AC = \frac{1-2}{3-8} = \frac{1}{5}$$
$$Gradient\ \perp AC = -5$$
$$Point\ (mid-point) of\ AB, M\left(\frac{3+2}{2}, \frac{1+6}{2}\right)$$
$$= M\left(\frac{5}{2}, \frac{7}{2}\right)$$
$$Point\ (mid-point) of\ AC, N\left(\frac{3+8}{2}, \frac{1+2}{2}\right) = N\left(\frac{11}{2}, \frac{3}{2}\right)$$

$$\boldsymbol{Equation\ of\ \perp bisector\ of\ AB}$$
$$y - y_1 = m(x - x_1)$$
$$y - \frac{7}{2} = \frac{1}{5}\left(x - \frac{5}{2}\right)$$
$$\mathbf{5y - x = 15} \ldots\ldots\ldots\ldots\ldots\ldots\ldots..(\boldsymbol{i})$$

$$\boldsymbol{Equation\ of\ \perp bisector\ of\ AC}$$
$$y - \frac{3}{2} = -5\left(x - \frac{11}{2}\right)$$
$$2y + 10x = 58$$
$$\boldsymbol{y + 5x = 29} \ldots\ldots\ldots\ldots\ldots\ldots\ldots..(\boldsymbol{ii})$$

Solving the simultaneous

$$y + 5x = 29 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (ii)$$
$$\underline{5y - x = 15 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (i)}$$
$$y + 5x = 29$$
$$\underline{25y - 5x = 75}$$
$$26y = 104$$
$$\boldsymbol{y = 4 \text{ then } x = 4}$$
$$\therefore R(5,4) \text{ the coordinate of the centre of the circle.}$$
$$\text{then the radius}, r^2 = \overline{RA}^2 = \overline{RB}^2 = \overline{RC}^2$$

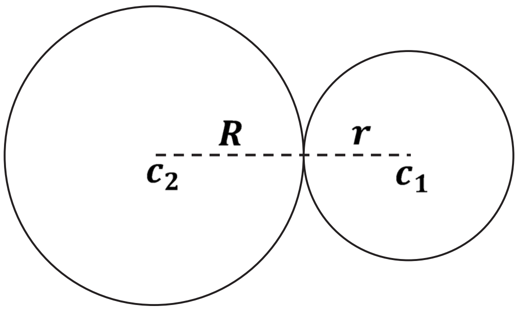
$\overline{RA} = \sqrt{13}.$

So

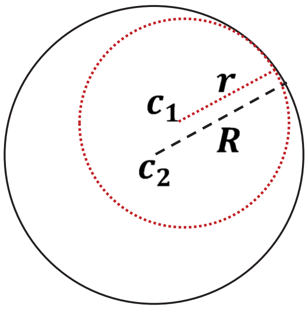
$$\text{From } (x-a)^2 + (y-b)^2 = r^2$$
$$(x-5)^2 + (y-4)^2 = 13$$
$$\text{Thus the equation of the circle.}$$

Circles that Touch Each Other

Two circles touch each other externally. If the sum of the radii is equal to the distance between their centres.



$$i.e.\ \boldsymbol{r + R = \overline{c_1c_2}}$$



If they touch internally then the difference between their radii is equal to the distance between their centres . i.e.

$$i.e. \boldsymbol{R - r = \overline{c_1c_2}}$$

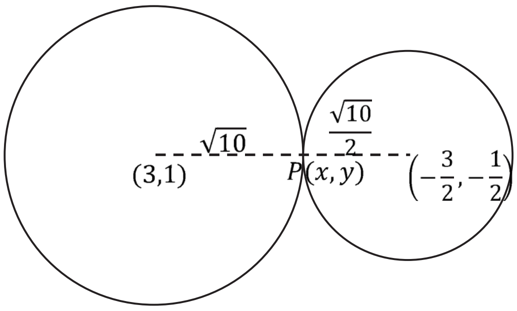
Prove that the circles $x^2 + y^2 + 3x + y = 0$ and $x^2 + y^2 - 6x - 2y = 0$ touch each other, find the coordinates of point of contact and equation of their common tangent at the point of contact.

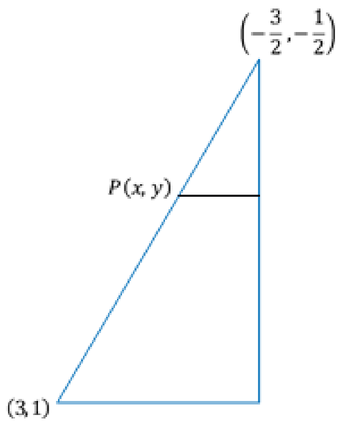
Solution

$$for\ \ x^2 + y^2 + 3x + y = 0$$
$$\left(x + \frac{3}{2}\right)^2 + \left(y + \frac{1}{2}\right)^2 = \frac{9}{4} + \frac{1}{4} = \frac{10}{4}$$
$$c_1\left(-\frac{3}{2}, -1/2\right) \text{ and } r = \frac{\sqrt{10}}{2}$$

$$for\ x^2 + y^2 - 6x - 2y = 0$$
$$(x-3)^2 + (y-1)^2 = 9 + 1 = 10$$
$$\boldsymbol{c_2}(3,1) \text{ and } r = \sqrt{10}$$
$$\overline{c_1c_2}^2 = (3 + 3/2)^2 + (1 + 1/2)^2 = \frac{81}{4} + \frac{9}{4} = \frac{90}{4}$$
$$\overline{\boldsymbol{c_1c_2}} = \frac{3}{2}\sqrt{10}, \qquad \boldsymbol{r + R} = \frac{\sqrt{10}}{2} + \sqrt{10} = \frac{3}{2}\sqrt{10}$$

Therefore, the two circles touch externally.





$$\frac{\left(x+\frac{3}{2}\right)}{3-x}=\frac{\frac{\sqrt{10}}{2}}{\sqrt{10}}=\frac{1}{2}$$
$$\Rightarrow \frac{2x+3}{6-2x}=\frac{1}{2}$$

$$\frac{y+\frac{1}{2}}{1-y}=\frac{1}{2}$$
$$\Rightarrow \frac{2y+1}{2-2y}=\frac{1}{2}$$
$$\Rightarrow 4y+2=2-2y$$
$$\Rightarrow y=0$$

The point of intersection $(\mathbf{0},\mathbf{0})$

The equation of tangent

$$Gradient\ of\ c_1c_2=\frac{1-0}{3-0}=\frac{1}{3}$$

Gradient of the tangent is -3.

The equation;

$$(y-0)=-3(x-0)$$
$$y=-3x\ or\ y+3x=0$$

The intersection of two circles

When the two equation of circles is solved simultaneously. We obtain the equation of common chord to the two circles.

Example

1. Find the equation of the common chord of the

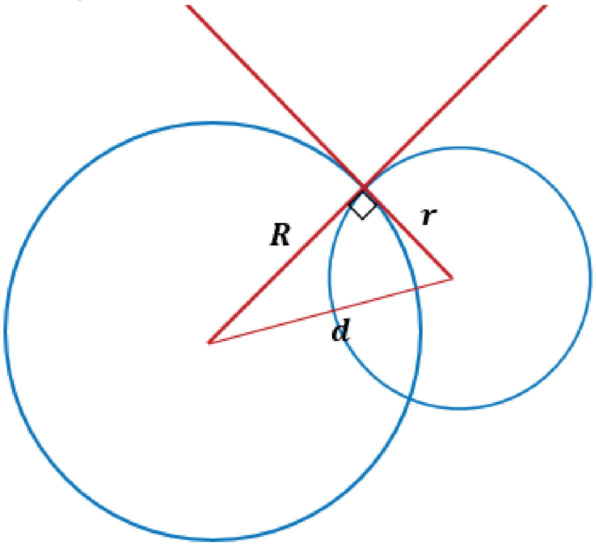
a. $x^2+y^2+3x+y=0$ and $x^2+y^2-6x-2y=0$

b. $x^2+y^2-4x-2y+1=0$ and $x^2+y^2+4x-6y-10=0$

Do exercise 21b page 410-411 (Pure Math Back House 1)

2. Show that the common chord to the circles $x^2+y^2=4$ and $x^2+y^2+4x-2y-4=0$ passes through the origin.

Orthogonal circles



$$d^2 = R^2 + r^2$$

The two circles are orthogonal if the tangents at the point of intersections of the circle are perpendicular.

3. Show that the circles

$$x^2 + y^2 + 10x - 4y - 3 = 0$$
$$x^2 + y^2 - 2x - 6y + 5 = 0 \text{ are orthogonal.}$$

Solution

For $x^2 + y^2 + 10x - 4y - 3 = 0$

$$(x+5)^2 + (y-2)^2 - 5^2 - 2^2 - 3 = 0;\ (x+5)^2 + (y-2)^2 = 5^2 + 2^2 + 3$$
$$(x+5)^2 + (y-2)^2 = 32; Centre\ , C_1(-5,2), R = \sqrt{32}$$

For $x^2 + y^2 - 2x - 6y + 5 = 0$

$$(x-1)^2 + (y-3)^2 - 1^2 - 3^2 + 5 = 0;\ (x-1)^2 + (y-3)^2 = 1^2 + 3^2 - 5$$
$$(x-1)^2 + (y-3)^2 = 5; Centre\ , C_2(1,3), r = \sqrt{5}$$

Distance, $d = \overline{C_1C_2} = \sqrt{\{(1 - -5)^2 + (3-2)^2\}} = \sqrt{\{36 + 1\}} = \sqrt{37}$

Checking whether $d^2 = R^2 + r^2 = 32 + 5 = 37$

Therefore, the two circles are orthogonal.

4. Find the equation of a circle which passes through (1,1) and cuts orthogonally each of the circle $x^2 + y^2 - 8x - 2y - 16 = 0$ and $x^2 + y^2 - 4x - 4y - 1 = 0$.

Topic: 20: Coordinate Geometry III (Conics)

Sub-topic 1: Parabola

Learning Outcomes

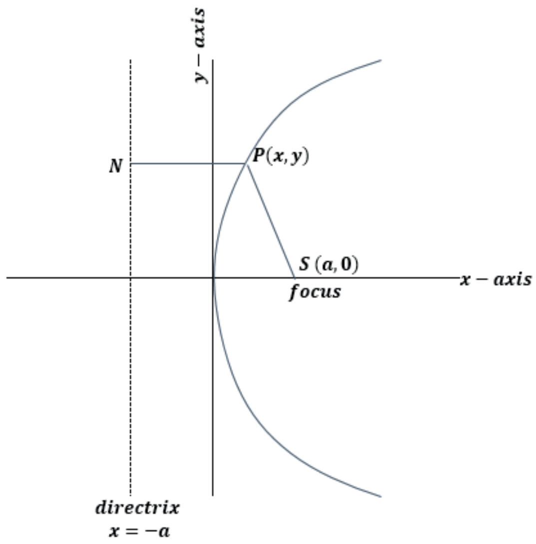
The learner should be able to:

i) identify the conics.
ii) identify a parabola.
iii) draw a sketch of a parabola and identify the equation of parabola.
iv) find the parametric equations of a parabola.
v) find the equation of tangent, normal and chord of a parabola.

Lesson: Parabola

The parabola is defined as the locus of those points equidistant from **a fixed point** and **a fixed straight line**. The fixed point is called the focus and the fixed straight line is called the directrix.

Suppose the line $x = -a$ is the directrix and the point $S(a, 0)$ is taken as the focus, then if $P(x, y)$ is a point on the parabola,



By definition; $PS = PN$

Hence $PS^2 = PN^2$

i.e. $(x - a)^2 + (y - 0)^2 = (x + a)^2$

which gives $y^2 = 4ax$ as the equation of the parabola.

Thus, Parabola $y^2 = 4ax$, focus $(a, 0)$, diretrix $x = -a$

So the parabola has the $x -$ $axis$ as a line of symmetry

Example 1

Sketch the graph of $(y - 2)^2 = 12(x - 1)$, showing clearly the focus and the directrix of the parabola.

This equation can be written $Y^2 = 12X$ where $Y = y - 2$ *and* $X = x - 1$.

Thus the curve $(y - 2)^2 = 12(x - 1)$ is a parabola with $a = 3$ and origin at $X = 0$, $Y = 0$ i.e. the point $(1,2)$

The parabola $y^2 = 4ax$ has origin$(0,0)$, focus $(a, 0)$ and directrix $x = -a$.

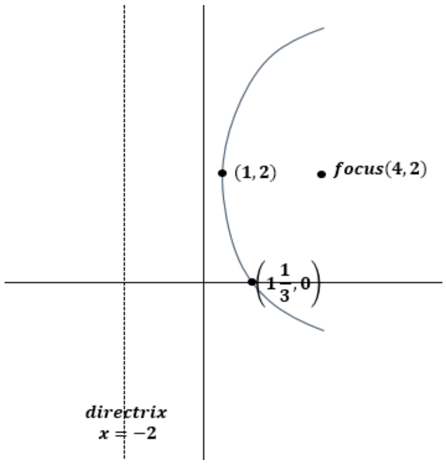
Thus with $a = 3$ and origin at $(1,2)$ the focus is at $(4,2)$ and directrix is $x = -2$.

When $x = 0$, $(y - 2)^2 = -12$, thus the curve does not cut the y- axis

When $y = 0$, $(-2)^2 = 12(x - 1)$, giving $x = 1\frac{1}{3}$.

Thus the curve cuts the $x - axis$ at $(1\frac{1}{3}, 0)$

Notice that the graph of $(y-2)^2 = 12(x-1)$ is that of $y^2 = 12x$ translated by 1 unit parallel to the $x-axis$ and 2 units parallel to the $y-axis$ which is consistent with the curve sketching.



Question 2. Show that the line $y = 3x + 1$ touches the parabola $y^2 = 12x$

(**Answer:** repeated roots $x = \frac{1}{3}$, $point\ \left(\frac{1}{3}, 2\right)$)

Gradient of a Particular Point

As with the circle, the gradient at a particular point on a parabola can be determined by implicit differentiation.

Question 3. Show that the point $A(2,-4)$ lies on the parabola $y^2 = 8x$ and find the equation of the normal to the parabola at the point A (Ans: $y = x - 6$)

Question 4. Find the equations of the tangents drawn from the point (1,3) to the parabola $y^2 = -16x$.

Suppose the tangents have equations of the form $y = mx + c$. These tangents pass the point (1,3), hence. $3 = m(1) + c \rightarrow 3 = m + c$ ……………1

Substituting for y from $y = mx + c$ into $y^2 = -16x$ gives $m^2x^2 + 2x(mc + 8) + c^2 = 0$

Now as $y = mx + c$ is a tangent to the curve$y^2 = -16x$, this quadratic must have a repeated root i.e. $b^2 = 4ac = 0$,

$\therefore 4(mc + 8)^2 - 4m^2c^2 = 0$, giving $m = -\frac{4}{c}$

Substituting this value of m in equation 1 gives $3 = -\frac{4}{c} + c$

I.e. $c^2 - 3c - 4 = 0$ giving $c = 4, m = -1\ or\ c = -1,\ m = 4$

Thus the required tangents have equations $y = -x + 4\ and\ y = 4x - 1$.

Parametric form

The general equation of the parabola,$y^2 = 4ax$, may be expressed in parametric form as $x = at^2, y = 2at$; Where t is the parameter. If we eliminate $'t'$ between these equations, we get $x =$

$a\left(\frac{y}{2a}\right)^2$ $or\ y^2 = 4ax$, it follows that the point $(at^2, 2at)$ lies on the parabola for values of t. Therefore, and for value of t there is one and only one point on the parabola.

Question 5

Find the equation of the normal to the parabola $y^2 = 4ax$ at the point $(at^2, 2at)$.

With $x = at^2$, $then \frac{dx}{dt} = 2at$ and $y = 2at\ then \frac{dy}{dt} = 2a$

Using $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = 2a.\frac{1}{2a} = \frac{1}{t}$

Thus the gradient of the tangent at $(at^2, 2at)$ is $\frac{1}{t}$ and therefore the gradient of the normal at this point is $-t$.

Using $y - y_1 = m(x - x_1)$ the equation of the normal at $(at^2, 2at)$ is $y = 2at = -t(x - at^2)$

I.e. $y + xt = 2at + at^3$ is the equation of the normal at the point $(at^2, 2at)$.

Question 6

The point $T(at^2, 2at)$ lies on the parabola $y^2 = 4ax$ and L is the point $(-a, 2a)$. M is the midpoint of TL. Find the equation of the locus of M as T moves on the parabola

Solution

M will have coordinates$\left(\frac{at^2-a}{2}, \frac{2at+2a}{2}\right)$.

As T moves on the parabola the parameter t varies and the parametric equations of the locus of M will be $x = \frac{a}{2}(t^2 - 1), y = a(t + 1)$. Eliminating t from these equations gives $y = 4(x + y)$ the Cartesian equation of the locus of M.

Questions

1. Given that the parabola $y^2 = 4ax$ has focus at $(a, 0)$. Write down the coordinates of the foci of the following parabolas;
    a) $y^2 = 4x$ ans; $(1, 0)$
    b) $y^2 = -8x$ ans; $(-2, 0)$
    c) $y^2 = 20x$ ans; $(5, 0)$
    d) $y^2 = 9x$ ans; $\left(2\frac{1}{4}, 0\right)$
2. Given that the parabola $y^2 = 4ax$ has directrix $x = -a$. Write down the equation of the directrix of each of the following parabolas.
    a) $y^2 = 12x$ Ans; $x = -3$
    b) $y^2 = -12x$ ans; $x = 3$
    c) $y^2 = 20x$ ans; $x = -5$
    d) $y^2 = -2x$ ans; $x = \frac{1}{2}$
3. Sketch the following parabolas showing clearly the focus and directrix of each one
    a) $(y - 2)^2 = 4(x - 3)$ $focus(4,2)$ $directrix;\ x = 2$ (4,0)
    b) $(y + 2)^2 = 8(x - 1)$ $focus(3, -2)$ $directrix;\ x = -1$ $(1\frac{1}{2}, 0)$
    c) $y^2 + 8y = 4x - 12 focus(0, -4); directrix;\ x = -2$; $(0, -2), (0, -6)$

4. Prove that the line $y = 2x + 2$ touches the parabola $y^2 = 16x$ and find the coordinates of the point where this occurs (ans$(1, 4)$)
5. Prove that the line $2y + 1 = 4x$ touches the parabola $y^2 + 4x = 0$ and find the coordinates of the point where this occurs. $\left(-\frac{1}{4}, -1\right)$

Sub-topic 2: Ellipse and Hyperbola

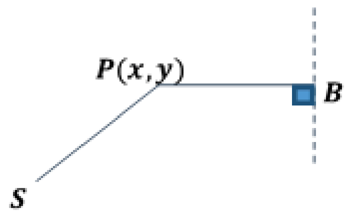
Learning Outcomes

The learner should be able to:

i) identify and sketch them.
ii) derive the general equation.
iii) determine the equation of tangent and normal to the ellipse or hyperbola at a given point.
iv) determine the parametric equations of an ellipse and the hyperbola.
v) write the equations of all the asymptotes of an ellipse and a hyperbola.

Lesson: The ellipse and the hyperbola

The parabola was defined as being the locus of points equidistant from a fixed point, the focus, and a fixed line, the directrix, i.e. those points $P(x, y)$ for which $PS = PB$.
If instead of PS and PB being equal, we have the two lengths in a constant ratio, say e: 1, the locus of all such points $P(x, y)$ will give the conic sections.
If $\frac{PS}{PB} = e$, called the eccentricity, then for $e = 1$, the locus of P is a parabola,
For $0 < e < 1$ the locus of P is an ellipse
For $e > 1$, the locus of P is a hyperbola.

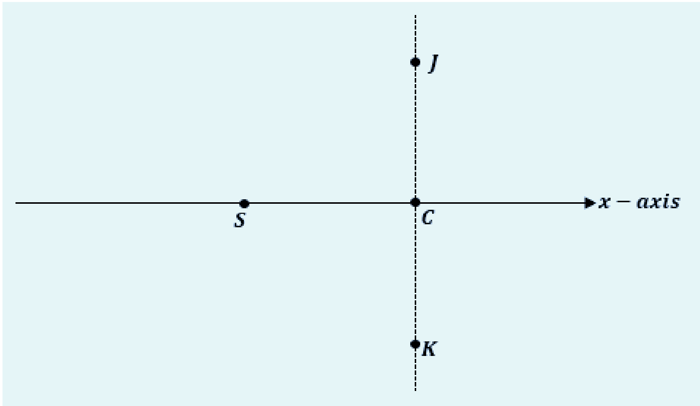


The ellipse

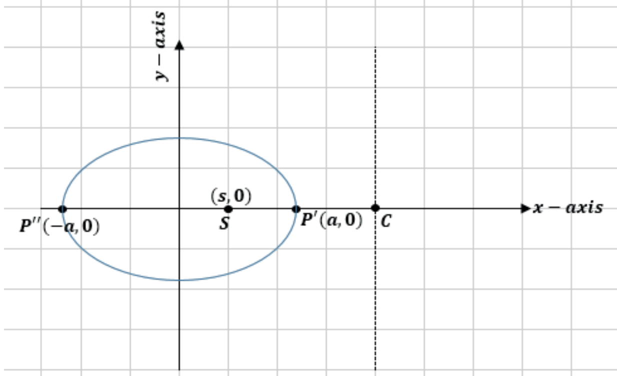
Suppose that S is the focus and the line JK the directrix. Take the x-axis as passing through S, perpendicular to JK. Suppose that JK cuts the $x - axis$ at C.

We require all points $P(x, y)$ that are such that

$$\frac{distance\ from\ P\ to\ S}{distance\ from\ P\ to\ JK} = e{:}\,1$$



Two such points p' and p'' will lie on the $x - axis$ with p' dividing SC internally in the ratio $e:1$ and p'' dividing SC externally in the ratio $e:1$.
Taking the origin as the midpoint of $p'p''$. We say p' has coordinates $(a, 0)$ and p''$(-a, 0)$. S is the point $(s, 0)$ and JK is the line $x = k$
Thus $SP' = e \times P'C$ gives $a - s = e(k - a)$
and
$SP'' = e \times P''C$ gives $a + s = e(k + a)$



Adding these equations gives $k = \frac{a}{e}$, and
subtracting these equations $s = ae$
Taking $P(x, y)$ as the general point on the ellipse

$$PS = ePB$$

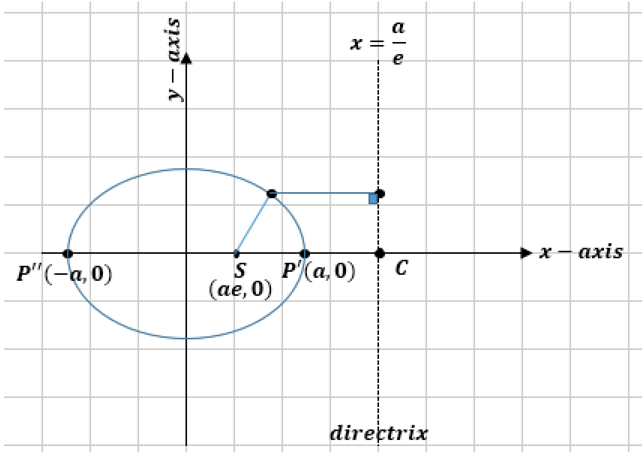
$\therefore PS^2 = e^2 PB^2$

$$\therefore (x - ae)^2 + (y - 0)^2 = e^2\left(\frac{a}{e} - x\right)^2$$

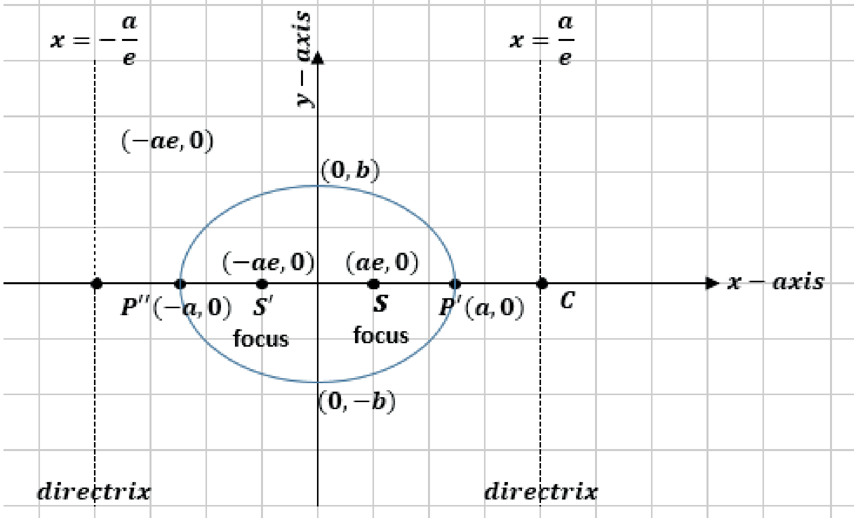
$$x^2 - 2xae + a^2e^2 + y^2 = a^2 - 2aex + x^2e^2$$

$$(1 - e^2)x^2 + y^2 = a^2(1 - e^2)$$

Thus $\frac{x^2}{a^2} + \frac{y^2}{a^2(1-e^2)} = 1$, writing $b^2 = a^2(1 - e^2)$, this gives $\boldsymbol{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$



Note that writing $(-x)$ for x and $(-y)$ for y in this equation leaves it unaltered.



Summary

For the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, b^2 = a^2(1 - e^2), e < 1,$ *foci at* $(\pm ae, 0)$ *directrices* $x = \pm\frac{a}{e}$.

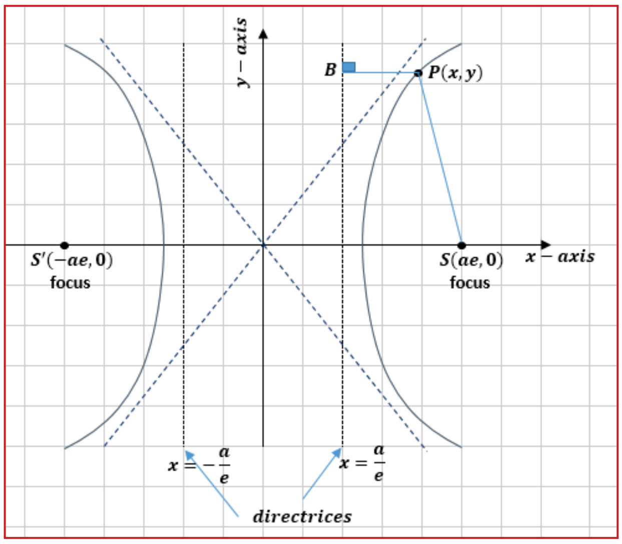
The hyperbola

In a similar way we can show that the hyperbola is as below

By definition

$PS = ePB \, , e > 1$

$\therefore PS^2 = e^2 PB^2$



$$\therefore (x - ae)^2 + (y - 0)^2 = e^2 \left(x - \frac{a}{e}\right)^2$$

$$x^2 - 2xae + a^2e^2 + y^2 = x^2e^2 - 2aex + a^2$$

$$(e^2 - 1)a^2 = x^2(e^2 - 1) - y^2$$

$1 = \frac{x^2}{a^2} - \frac{y^2}{a^2(e^2-1)}$ Writing $b^2 = a^2(e^2 - 1)$ gives

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Summary

For the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \; b^2 = a^2(e^2 - 1),$ $e > 1$ foci at $(\pm ae, 0)$ directrices $x = \pm\frac{a}{e}$

Notes:

1. In the diagram of the hyperbola, the lines to which the curve tends at infinity are shown and these are the asymptotes of the hyperbola.

Writing the equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ as $y^2 = b^2\left(\frac{x^2}{a^2} - 1\right)$ i.e. $y = \pm\frac{bx}{a}\sqrt{\left(1 - \frac{a^2}{x^2}\right)}$

As $x \to \infty, \frac{a^2}{x^2} \to 0$ *and* $y = \pm\frac{bx}{a}$ which are the equations of the asymptotes

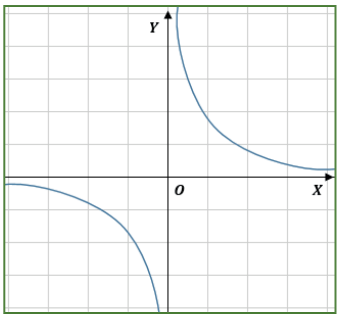
2. The special case when $e = \sqrt{2}$ gives $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and $b^2 = a^2\left(\left(\sqrt{2}\right)^2 - 1\right)$ *or* $b^2 = a^2$

$x^2 - y^2 = a^2$……………………………………1

Furthermore, the asymptotes are $y = x$ *and* $y = -x$

And are therefore perpendicular to each other.

Using the asymptotes as the coordinate axes it can be shown that equation 1 becomes $xy = c^2$. The curve is a <u>rectangular hyperbola</u> as shown above



Question 1

Find the equation of the tangent to the curve $4x^2 + 9y^2 = 36$ at the point $\left(1, \frac{4\sqrt{2}}{3}\right)$.

Differentiating with respect to x

$$8x + 18y\frac{dy}{dx} = 0, \quad \rightarrow \frac{dy}{dx} = -\frac{4x}{9y}$$

$$\therefore \textit{at the point } \left(1, \frac{4\sqrt{2}}{3}\right) \textit{ the gradient} = -\frac{4(1)}{9\left(\frac{4\sqrt{2}}{3}\right)} = -\frac{1}{3\sqrt{2}}$$

Thus the gradient of the tangent at $\left(1, \frac{4\sqrt{2}}{3}\right)$ is $-\frac{1}{3\sqrt{2}}$using $y - y_1 = m(x - x_1)$, the equation of the tangent is $y - \frac{4\sqrt{2}}{3} = -\frac{1}{3\sqrt{2}}(x-1)$ giving $3\sqrt{2} \times y + x = 9 \rightarrow 3y\sqrt{2} + x = 9$

Question 2

Find the coordinates of the point at which the normal to the curve $xy = 8$, at the point (4,2), cuts the tangent to the curve $16x^2 - y^2 = 64$ at the point$\left(2\frac{1}{2}, 6\right)$.

$$xy = 8, \quad y + x\frac{dy}{dx} = 0, \quad \rightarrow \frac{dy}{dx} = -\frac{y}{x}$$

At the point (4,2) the gradient of the tangent is $-\frac{2}{4} = -\frac{1}{2}$

$$\therefore \textit{Gradient of normal at } (4,2) \textit{is } 2,$$

$$\textit{Equation of normal is } \; y - 2 = 2(x-4), \;\; y = 2x - 6 \ldots\ldots\ldots\ldots\ldots\ldots..1$$

$$16x^2 - y^2 = 64, \quad 32x - 2y\frac{dy}{dx} = 0 \;\; \rightarrow \frac{dy}{dx} = \frac{16x}{y}$$

At the point (2 ½, 6) , the gradient of the tangent is $\frac{16}{6}\left(2\frac{1}{2}\right) = \frac{20}{3}$

$$\textit{Equation of tangent is } y - 6 = \frac{20}{3}\left(x - \frac{5}{2}\right), i.e. \;\; 3y = 20x - 32 \ldots\ldots\ldots\ldots.2$$

Solving equations 1 and 2 simultaneously gives $x = 1$, $y = -4$.

Therefore, thus the required normal and tangent intersect at the point $(1, -4)$

Parametric forms

For the ellipse; $\boldsymbol{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$**,**

We use the parameter θ and $\cos^2\theta + \sin^2\theta = 1$.

We can give the equation as $x = a\cos\theta$, $y = b\sin\theta$

Thus the general point on the ellipse has parametric coordinates ($a\cos\theta$, $b\sin\theta$)

For the hyperbola; $\boldsymbol{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$**,**

We again use the parameter θ and since $\sec^2\theta - \tan^2\theta = 1$

We can give the equation as $x = \text{a}\sec\theta$, $y = b\tan\theta$.

Thus the general point on the hyperbola has parametric coordinates ($a\sec\theta$, $b\tan\theta$)

For the **rectangular hyperbola;**$\boldsymbol{xy = c^2}$, we use the parameter $'t'$ and the equation may then be written as $x = ct$, $y = \frac{c}{t}$

The parametric coordinates of the general point on the rectangular hyperbola are then $\left(ct, \frac{c}{t}\right)$

Question 3. Find the equation of the tangent to the ellipse $\boldsymbol{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$ at the point ($a\cos\theta$, $b\sin\theta$)

$$x = a\cos\theta \;\; , \frac{dx}{d\theta} = -a\sin\theta \;\; \textit{and } y = b\sin\theta, \frac{dy}{d\theta} = b\cos\theta$$

$$using\frac{dy}{dx}=\frac{dy}{d\theta}\times\frac{d\theta}{dx}, \quad \frac{dy}{dx}=b\cos\theta\ .-\frac{1}{a\sin\theta}=-\frac{b\ cos\ \theta}{a\sin\theta}$$

Using $y-y_1=m(x-x_1)$ the equation of the tangent

Is $y-b\sin\theta=-\frac{bcos\ \theta}{\text{asin}\ \theta}(x-a\cos\theta)$

$$\therefore ay\sin\theta-ab\sin^2\theta=-bx\cos\theta+ab\cos^2\theta\ \ ,\rightarrow aysin\ \theta+bxcos\ \theta=ab$$

The equation of the tangent at $(\ a\cos\theta\ , b\sin\theta\ )$ on the ellipse $\boldsymbol{\frac{x^2}{a^2}+\frac{y^2}{b^2}}=\mathbf{1}$ is

$$aysin\ \theta+bxcos\ \theta=ab$$

Term 3

Topic 21: Complex Numbers

Learning Outcomes

The learner should be able to:

i) identify a complex number.
ii) simplifying powers of *I.*
iii) solve quadratic equations having imaginary roots.
iv) identify the real and imaginary parts of complex numbers.
v) identify and state a conjugate of a complex number.

Sub-topic 1: Imaginary Numbers

Learning Outcomes: to learn about complex numbers.

Lesson: Imaginary numbers

Considering an equation

Say $\boldsymbol{x^2 + 1 = 0}$ or $\boldsymbol{x^2} = \pm\sqrt{-\mathbf{1}}$ has no real roots.

Instead we write $\boldsymbol{i}$ for $\sqrt{-\mathbf{1}}$, so the roots of $x^2 + 1 = 0$, $\boldsymbol{x = \pm i}$

Also $\sqrt{(-25)} = \sqrt{25 \times -1} = \sqrt{25} \times \sqrt{-1} = 5i$

Solve the equations

$$x^2 + 64 = 0 \quad , x^2 + 7 = 0 \quad (x+3)^2 = -25$$

So an equation in the form

$$x^2 + n^2 = 0 \quad or\ x^2 = -n^2 \quad where\ \ n \in \mathbb{R}, has\ two\ roots, x = \pm\, ni$$

Complex numbers

1. **Solving $\boldsymbol{ax^2 + bx + c = 0}$when $\boldsymbol{b^2 < 4ac}$.**

   Example

$$x^2 - 4x + 5 = 0, \Rightarrow x^2 - 4x = 5, \Rightarrow (x-2)^2 = 4 - 5, \Rightarrow (x-2)^2 = -1$$

$$x - 2 = \pm\sqrt{-1}\,, \Rightarrow x - 2 = \pm i,$$

$$\boldsymbol{Either\ x = 2 + i\ or\ x = 2 - i}$$

---

2. **Solve $\boldsymbol{x^2 - 6x + 34 = 0}$**

   Using

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4 \times 1 \times 34}}{2 \times 1} = \frac{6 \pm \sqrt{36 - 136}}{2} = \frac{6 \pm \sqrt{-100}}{2} = \frac{6 \pm i\sqrt{100}}{2} = \frac{6 \pm 10i}{2}$$

$$x = \frac{6 \pm 10i}{2} = \frac{2(3 \pm 5i)}{2} = 3 \pm 5i, \quad Either\ x = 3 + 5i\ \ or\ \ x = 3 - 5i$$

Numbers of the form$\boldsymbol{p + iq}$, where $\boldsymbol{p}$ and $\boldsymbol{q}$ are real numbers are called ***complex numbers***
Where $\boldsymbol{p}$ is called the **real part** of the complex number, and$\boldsymbol{q}$is called the **imaginary part**.
Considering the complex number. $\boldsymbol{z = p + iq}$, $Re(\boldsymbol{z}) = \boldsymbol{p}$ $and\ Im(\boldsymbol{z}) = \boldsymbol{q}$

$$E.g.\ \boldsymbol{z = 2 + 7i},\ \ Re(z) = 2\ \ and\ Im(z) = 7$$

Solve the quadratic

a) $z^2 - 4z + 13 = 0$
b) $9z^2 + 25 = 0$
c) $2z^2 = 2z - 13$
d) $34z^2 - 6z + 1 = 0$

Note: The general formula: $\boldsymbol{z} = \frac{\boldsymbol{-b \pm i\sqrt{(4ac - b^2)}}}{\boldsymbol{2a}}$

Sub-topic: 2: Algebra of Complex Numbers

Learning Outcomes

The learner should be able to:

i) add, subtract, multiply and divide complex numbers.
ii) solve unknowns by:
- comparing coefficients.
- using sum and product of root for quadratics.

iii) formulate equations using complex roots.
iv) use of the identity $(a^3 \pm b^3) = (a \pm b)(a^2 \pm ab + b^2)$in finding roots of real numbers.

Lesson:

The algebra of imaginary numbers

a) Addition and Subtraction

$2i + 6i = 8i$ | $\sqrt{8}i + \sqrt{2}i = 3\sqrt{2}i$ | $10i - 6i = 4i$ | $13i - 7i + 6i = 12i$

b) Multiplication of imaginary numbers

i. $3i \times 2i = 6.i^2 = -6$
ii. $10i \times 2i \times 3i = 60i^3 = 60(i^2).i = -60i$
iii. $i^4 = (i^2)^2 = (-1)^2 = 1$
iv. $i^{10} = (i^2)^5 = (-1)^5 = -1$
v. $i^{-1} = \frac{1 \times i}{i \times i} = \frac{i}{i^2} = \frac{i}{-1} = -i$

Algebra of complex numbers

a) Addition and subtraction

When adding or subtracting two complex numbers a real number is added or subtracted to a real number and imaginary to imaginary.

i.e.

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$
$$(a + bi) - (c + di) = (a - c) + (b - d)i$$

b) Multiplication of complex number

i. $(3 + 4i) \times (2 + 6i) = 3(2 + 6i) + 4i(2 + 6i) = 6 + 18i + 8i + 24i^2$

$$= 6 + 26i - 24$$
$$= -18 + 26i$$

And

ii. $(5 + 6i)(5 - 6i) = 25 - 30i + 30i - 36i^2 = 25 + 36 = 61$

Note:

The product is a **real number**. This is because of the special form of the given complex $\mathbf{5 \pm 6i}$.

Any pair of complex numbers of the form $\boldsymbol{a \pm bi}$ have a product which is real since $(\boldsymbol{a + bi})(\boldsymbol{a - bi}) \equiv \boldsymbol{a^2 + b^2}$ Such complex numbers are said to be conjugate of the other. If $\boldsymbol{z = a + bi}$ then the conjugate is $\boldsymbol{\bar{z} = a - bi}$. It's represented by $\boldsymbol{z^*}$ ***or*** $\boldsymbol{\bar{z}}$

$$But\ \ (\boldsymbol{a + bi})(\boldsymbol{c + id}) = (\boldsymbol{ac - bd}) + \boldsymbol{i}(\boldsymbol{ad + bc})$$

c) Division

Division of two complex numbers cannot be carried out because the denominator contains two terms which cannot be simplified, therefore, Division is carried out when the denominator is made real i.e. by multiplying the numerator and denominator and the process is called ***realizing the denominator***.

i. $\frac{2}{5i} = \frac{2}{5i} \times \frac{i}{i} = \frac{10i}{25i^2} = -\frac{2i}{5}$

ii. $\frac{3+4i}{7+6i} = \frac{3+4i}{7+6i} \times \frac{7-6i}{7-6i} = \frac{3(7-6i)+4i(7-6i)}{7^2-36^2i^2} = \frac{21-18i+28i+24}{49+36} = \frac{45+10i}{85} = \frac{45}{85} + \frac{10}{85}i = \frac{9}{17} + \frac{2}{17}i$

Definition

Two complex numbers in the form $\boldsymbol{x + iy}, \boldsymbol{x - iy}$ are called **conjugate complex numbers**. The symbol $\boldsymbol{z^*}$ *or* $\boldsymbol{\bar{z}}$ is used.

If $\boldsymbol{z = x + iy}$ $, \boldsymbol{z^* = x - iy}$ ***or*** $\boldsymbol{\bar{z} = x - iy}$

Example 1: If $\alpha\ and\ \beta$ are the roots of $\boldsymbol{z^2 - 10z + 29 = 0}$. find $\alpha\ and\ \beta$ by using the formula. Verify that $\boldsymbol{\alpha + \beta = 10}$ *and* $\boldsymbol{\alpha\beta = 29}$.

Solving for the two roots

Using $\boldsymbol{z} = \frac{\boldsymbol{-b \pm i\sqrt{(4ac - b^2)}}}{\boldsymbol{2a}}$

Where $a = 1, b = -10 \; and \; c = 29$

$$z = \frac{-\,^{-}10 \pm i\sqrt{\{(4 \times 1 \times 29) - (-10)^2\}}}{2 \times 1} = \frac{10 \pm 4i}{2} = 5 \pm 2i$$

$$Therefore, \alpha = 5 + 2i \; and \; \beta = 5 - 2i$$

Verifying

$$\alpha + \beta = 5 + 2i + 5 - 2i = 10$$

And $\alpha \times \beta = (5 + 2i)(5 - 2i) = 25 + 4 = 29$

Hence verified.

Example 2: If $\alpha \; and \; \beta$ are the roots of $\boldsymbol{az^2 + bz + c = 0}$. find, by using the formula, expressions for $\alpha \; and \; \beta$ in terms of $a, b \; and \; c$. Verify that $\boldsymbol{\alpha + \beta = -\frac{b}{a}} \; and \; \boldsymbol{\alpha\beta = \frac{c}{a}}$

Example 3: Solve the cubic equation $2z^3 + 3z^2 + 8z - 5 = 0$.

By inspection, if $f(z) = 2z^3 + 3z^2 + 8z - 5, if \; z = 0.5, f(0.5) = 0$

So it's a root $\boldsymbol{z - \frac{1}{2} = 0}$

Using long division. Ans. $\frac{1}{2}, -1 + 2i \; and -1 - 2i$

$$\begin{array}{r} 2z^2 + 4z \; + 10 \\ (z - 0.5)\overline{)\,2z^3 + 3z^2 + 8z - 5} \\ \underline{2z^3 - z^2} \qquad\qquad \\ 4z^2 + 8z \qquad \\ \underline{4z^2 - 2z} \qquad \\ 10z - 5 \\ \underline{10z - 5} \end{array}$$

Therefore, $f(z) = 2z^3 + 3z^2 + 8z - 5 = (z - 0.5)(2z^2 + 4z \; + 10)$

Solving $2z^2 + 4z \; + 10 = 0$

$$\boldsymbol{z = \frac{-4 \pm i\sqrt{(4 \times 2 \times 10 - 4^2)}}{2 \times 2} = \frac{-4 \pm 8i}{4} = 1 \pm 2i}$$

Therefore the roots are $\frac{1}{2}, -1 + 2i \; and -1 - 2i$

Example 4: Prove that $3x - 2$ is a factor of $3x^3 - 2x^2 + 3x - 2$, find the solution set of the equation $3x^3 - 2x^2 + 3x - 2 = 0$. when x belongs to the set of

a) Integers Z b) rational numbers R c) real numbers R d) complex numbers C.

1. Find the square root of $-5 + 12i$ by equating real and imaginary parts of

$$(a + ib)^2 = -5 + 12i$$

$$(a + ib)^2 = a^2 - b^2 + 2abi = -5 + 12i$$

$$Real: a^2 - b^2 = -5$$

$$Imaginary: 2ab = 12 \; \rightarrow \; ab = 6, a = \frac{6}{b}$$

$$\frac{36}{b^2} - b^2 = -5 \; \rightarrow 36 - b^4 = -5b^2, \;\; b^4 - 5b^2 - 36 = 0$$

$$b^2 = \frac{5 \pm \sqrt{(25 - 4 \times -36)}}{2} = \frac{5 \pm 13}{2}$$

$$Either\ b^2 = \frac{5+13}{2} = 9\ ;\ b = \pm 3$$

$$or\ b^2 = \frac{5-13}{2} = -4\ ;b = \pm 2i$$

For $when\ b = 3, a = \frac{6}{3} = 2\ or\ when\ \ b = -3, a = \frac{6}{-3} = -2; i.e. 2 + 3i\ and - 2 - 3i$

For $when\ b = 2i, a = \frac{6}{2i} = -3i\ or\ when\ \ b = -2i, a = \frac{6}{-2i} = 3i;$

$$i.e. -3i + (2i)i = -2 - 3i\ and 3i + (-2i)i = 2 + 3i$$

Note complete using one set of values.

Sub-topic 3: Argand Diagram and Polar Form

Learning Outcomes

The learner should be able to:

i) find the modulus and argument of a complex number.
ii) represent complex numbers on an Argand diagram.
iii) express complex numbers in terms of polar coordinates.
iv) express complex numbers in the polar form.

Lesson: Complex number as ordered pair

We know that complex number is of the form $\boldsymbol{z = x + iy}$ where$\boldsymbol{x}$ and $\boldsymbol{y}$ are real numbers. We may represent $\boldsymbol{z = x + iy}$ by $(\boldsymbol{x}, \boldsymbol{y})$or $\begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{pmatrix}$

i.e. $z_1(2,3), z_2(1,4) z_1 + z_2 = (2 + 1\ , 3 + 4) = (3,7) = 3 + 7i$

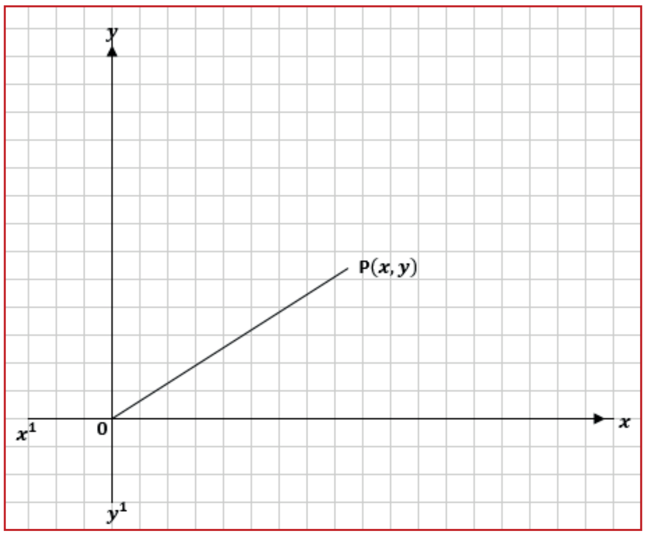
Geometrical representation: The Argand diagram

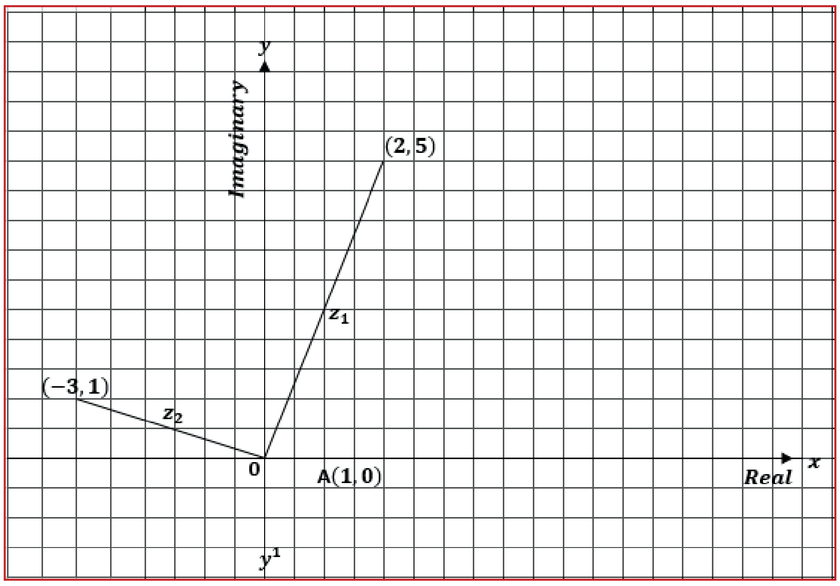
Let **XOX'** and **YOY'** be the coordinates axes and Origin, then any complex **number** $\boldsymbol{z = x + iy = (x, y)}$.

The representation of the complex numbers as points in the plane forms the argand diagram.

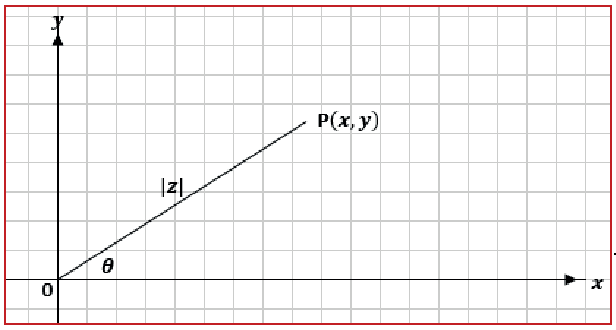
In the argand the real term is represented on the $x - axis$ and the imaginary on the $y - axis$.

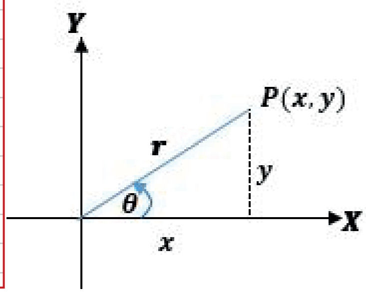
I.e. represent the following on an argand diagram.$z_1 = 2 + 5i, z_2 = -3 + i, z_3 = 4i$





Modulus of arguments of a complex numbers
The modulus or length of complex number $z = x + yi$





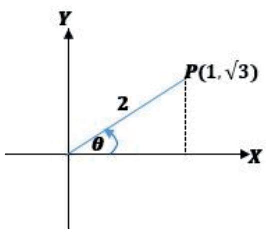
$$|z| = \overline{OP} = \sqrt{x^2 + y^2}$$

I.e. find the modulus of $z_1 = 3 + 4i$, $|z_1| = \sqrt{3^2 + 4^2} = 5$
the angle between the **positive x** and **line OP** is called the **argument of z** written as $\boldsymbol{Arg(z)}$
The unique value of θ such that $-\boldsymbol{\pi} \leq \boldsymbol{\theta} \leq \boldsymbol{\pi}$ is called the principle argument

Note:
Before finding the $\boldsymbol{Arg(z)}$it is advisable to first represent $\boldsymbol{z}$ on an argand diagram such that you can actually see in which quadrant θ lies. $\boldsymbol{Arg(z)} = \mathbf{tan^{-1}}\left(\frac{y}{x}\right)$
Find the modulus and argument of the following complex numbers.

i. $z_1 = 1 + \sqrt{3}i$

$$|z_1| = \sqrt{\left((1)^2 + (\sqrt{3})^2\right)} = \mathbf{2} \& \arg(z_1) = \tan^{-1}(\sqrt{3})$$
$$= \frac{\boldsymbol{\pi}}{\mathbf{3}}$$



ii. $z_2 = -1 + \sqrt{3}i$

$$|z_2| = \sqrt{\left((-1)^2 + \left(\sqrt{3}\right)^2\right)} = \mathbf{2}\&\arg(z_2) = \tan^{-1}\left(-\sqrt{3}\right) = \pi - \tan^{-1}\left(\sqrt{3}\right) = \frac{\mathbf{2\pi}}{\mathbf{3}}$$

iii. $z_3 = -1 - \sqrt{3}i$

$$|z_3| = \sqrt{\left((-1)^2 + \left(-\sqrt{3}\right)^2\right)} = \mathbf{2}\&$$

$$\arg(z_3) = \tan^{-1}\left(\frac{-\sqrt{3}}{-1}\right) = -\left(\pi - \tan^{-1}\left(\sqrt{3}\right)\right) = -\frac{\mathbf{2\pi}}{\mathbf{3}}$$

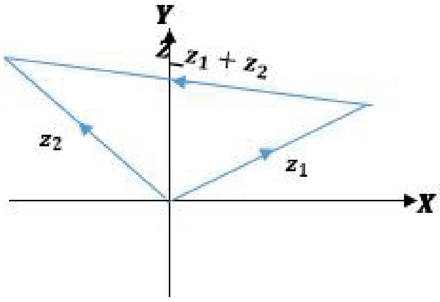
iv. $z_4 = 1 - \sqrt{3}i$

$$|z_4| = \sqrt{\left((1)^2 + \left(-\sqrt{3}\right)^2\right)} = \mathbf{2}\&$$

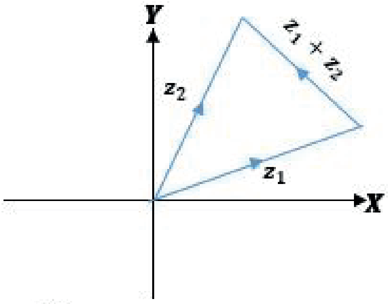
$$\arg(z_4) = \tan^{-1}\left(\frac{-\sqrt{3}}{1}\right) = -\tan^{-1}\left(\sqrt{3}\right) = -\frac{\boldsymbol{\pi}}{\mathbf{3}}$$

Geometric representation of

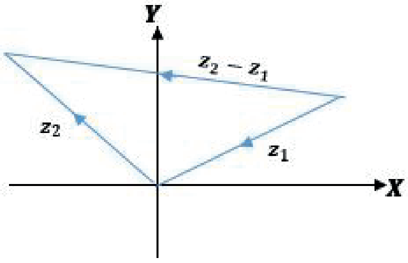
i. $z_1 + z_2$



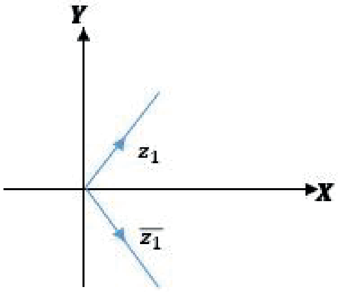
OR



ii. $z_2 - z_1$

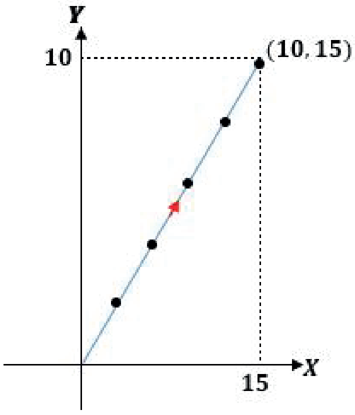


iii. $z_1 \; and \; {z_1}^*$



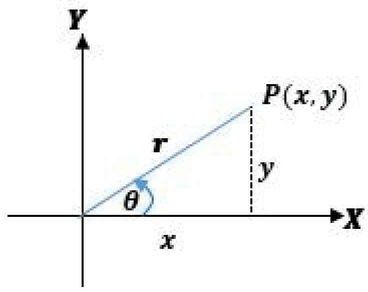
iv. $5z_1$

$$if\ z = 2 + 3i$$
$$5z = 5(2 + 3i) = 10 - 15z$$



Polar form of complex numbers

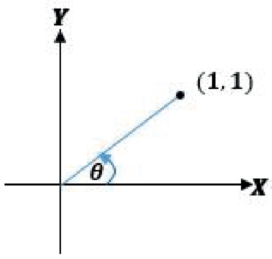
Consider a complex number $= x + yi$. Let$arg(z) = \theta, |z| = r$



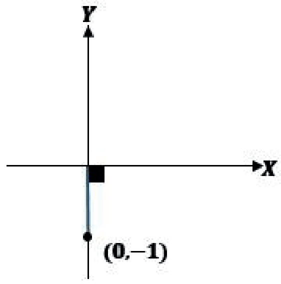
$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = r\cos\theta + i\, r\sin\theta$$
$$z = r(\cos\theta + i\sin\theta)$$

Express the following number in Polar form

I. $z = 1 + i$



II. $z = -i$



More examples

1. The equation $x^4 - 4x^3 + 3x^2 + 2x - 6 = 0$ has a root $1 - i$, find the other three roots.
2. Given that $1, w_1, w_2$ are the roots of the equation $z^3 = 1$ express $w_1 and\ w_2$ in the form $x + iy$ and hence or otherwise, show that
    a. $(i)\ 1 + w_1 + w_2 = 0$
    b. $(ii) \frac{1}{w_1} = w_2$
3. Solve each of the equations
    a. $(x + 4)(5x - 7) = 0$
    b. $(x^2 + 4)(5x^2 - 7) = 0$ when x belongs to the set of (a) integers, (b) rational numbers, (c) real numbers and (d) complex numbers.
4. Find the value of a and b such that $(a + ib)^2 = i$. Hence or otherwise solve the equation $z^2 + 2z + 1 - i = 0$, giving your answers in the form $p + iq$, where $p\ and\ q$ are real numbers.
5. (a) Given that the complex numbers w_1 and w_2 are the roots of the equation $z^2 - 5 - 12i = 0$, express w_1 and w_2 in the form $a + ib$, where a and b are real.
    (b) Indicate the point sets in an Argand diagram corresponding to the sets of complex numbers.
    $$A = \{z: |z| = 3, z \in \mathbb{C}\}$$
    $$B = \{z: |z| = 2, z \in \mathbb{C}\}$$
    Shade the region corresponding to values of z for which the inequalities $2 < |z| < 3$ and$30^0 < \arg z < 60^0$ are simultaneously satisfied.
6. Given that $z = 3 + 4i\ and\ w = 12 + 5i$, write down the moduli and arguments of
    a. z
    b. w

c. $\frac{1}{z}$
d. $\frac{1}{w}$
e. zw
f. z^*
g. w^*
h. $(zw)^*$
i. z^2
j. w^2

7. $Simplify$: $(1+i)^2$, $(1+i)^3$,$(1+i)^4$, $\frac{1}{3-2i}$, $\frac{1}{(1-i)^2}$. Draw in the Argand diagram the radius vectors corresponding to $(1+i)^2$, $(1+i)^3$,$(1+i)^4$. Find the principal values of the arguments of these complex numbers.

Sub-topic 4: Locus

Learning Outcomes

The learner should be able to:

i) find and define the locus of given complex equations and inequalities.
ii) describe and represent the locus on an Argand diagram.

Lesson:

The set of numbers

$$\mathbb{C} = \{x + iy: x, y \in \mathbb{R}\} \ where \ i^2 = -1$$

If $z_1 = x_1 + iy_1 \ and \ z_2 = x_2 + iy_2$ then

A. $z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$
B. $z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2)$
C. $z_1 \times z_2 = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$
D. $\frac{z_1}{z_2} = \frac{x_1+iy_1}{x_2+iy_2} = \frac{x_1+iy_1}{x_2+iy_2} \times \frac{x_1-iy_1}{x_2-iy_2} = \frac{(x_1x_2+y_1y_2)+i(x_2y_1-x_1y_2)}{{x_2}^2+{y_2}^2}$

if we write $|z| = r$ and $arg(z) = \theta$, then $|z_1z_2| = r_1r_2 = |z_1| \times |z_2|$ and

$$\arg(z_1z_2) = \theta_1 + \theta_2 = \arg(z_1) + \arg(z_2)$$

Similarly,

$$\left|\frac{z_1}{z_2}\right| = \frac{r_1}{r_2} = \frac{|z_1|}{|z_2|} \ and \ \arg\left(\frac{z_1}{z_2}\right) = \theta_1 - \theta_2 = \arg(z_1) - \arg(z_2)$$

The following special cases are important

a) $|z^2| = |z|^2 \ and \ \arg(z^2) = 2\arg(z)$

b) $\left|\frac{1}{z}\right| = \frac{1}{|z|} \ and \ \arg\left(\frac{1}{z}\right) = -\arg(z)$

For any complex numbers z_1 and z_2 prove that

i) $|z_1 z_2| = |z_1| \times |z_2|$ and $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

$$z_1 z_2 = r_1(\cos\theta_1 + i\sin\theta_1) r_2(\cos\theta_2 + i\sin\theta_2)$$

$$z_1 z_2 = r_1 r_2(\cos\theta_1\cos\theta_2 + i\sin\theta_2\cos\theta_1 + i\sin\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2)$$

$$z_1 z_2 = r_1 r_2(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i(\sin\theta_1\cos\theta_2 + \sin\theta_2\cos\theta_1)$$

$$z_1 z_2 = r_1 r_2(\cos(\theta_1+\theta_2) + \sin(\theta_1+\theta_2))$$

$$|z_1 z_2| = r_1 r_2 = |z_1| \times |z_2|$$

$$\arg(z_1 z_2) = \theta_1 + \theta_2 = \arg(z_1) + \arg(z_2)$$

ii) $$\left|\frac{z_1}{z_2}\right| = \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} \times \frac{(\cos\theta_1 - i\sin\theta_1)}{(\cos\theta_2 - i\sin\theta_2)} = \frac{r_1(\cos(\theta_1+\theta_2)+\sin(\theta_1+\theta_2))}{r_2(\cos^2\theta_2 + i\sin^2\theta_2)}$$

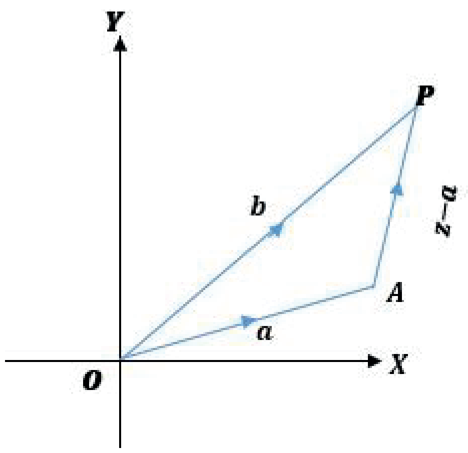
$$\left|\frac{z_1}{z_2}\right| = \frac{r_1}{r_2}(\cos(\theta_1 - \theta_2) + \sin(\theta_1 - \theta_2))$$

$$\left|\frac{z_1}{z_2}\right| = \frac{r_1}{r_2} = \frac{|z_1|}{|z_2|}$$

$$\arg\left(\frac{z_1}{z_2}\right) = \theta_1 - \theta_2 = \arg(z_1) - \arg(z_2)$$

Example: Given that $z_1 = -2 + 2i$ *and* $z_2 = \frac{1}{2} - \frac{\sqrt{3}}{2}i$, Express z_1 *and* z_2 in polar form hence find $|z_1 z_2|$ and $\arg(z_1 z_2)$

This analogy is very useful when interpreting complex numbers geometrically. In particular, diagrams like that is shown. Are very common.



We could regard this diagram either as an argand diagram illustrating the complex numbers z, a and $z - a$ or as a vector triangle representative vectors $\boldsymbol{z}, \boldsymbol{a}$ and $\boldsymbol{z} - \boldsymbol{a}$. Consequently, a statement in complex numbers, such as $|z - a| = r$

In which z represents a (variable) complex number $(x + iy)$, a is a constant complex number and r is given real constant, tells us that the length AP is equal to r; in other words the variable point P lies on a circle, centre A radius r.

Similarly a and b are complex numbers represented by fixed points A and B, tells us that the variable point P which represents Z is equidistant from A and B.

Example

1. Given that $z_1 = 3 + 4i$ *and* $z_2 = 1 + i$, find the modulus and argument of

$a)\ z_1 \qquad b) z_2 \qquad c) z_1 + z_2 \qquad d)\ z_1 - z_2 \qquad e) z_1 z_2 \qquad f) \frac{z_1}{z_2} \qquad g)\ (z_1)^2 \qquad h) \frac{1}{z_1}$

2. Writing $= x + iy$, find the equations of the following loci in terms of x and y

i. $|z - 10| = 5$

Since $z = x + iy$

So $|x + iy\ - 10| = 5$

Collect like term and find the modulus

$$|x + iy\ - 10| = 5;\ \ |(x - 10) + iy\,| = 5\,;$$
$$\sqrt{\{(x-10)^2 + y^2\}} = 5;$$
$$(x - 10)^2 + y^2 = 5^2$$

The locus is a circle of centre (10,0) and radius 5 units

ii. $|z - 1| = |z - i|$

Since $z = x + iy$

$$|x + yi - 1| = |x + yi - i|$$

Collecting like terms

$$|(x - 1) + yi| = |x + (y - 1)i|$$

Finding the moduli.

$$\sqrt{\{(x-1)^2 + y^2\}} = \sqrt{\{x^2 + (y-1)^2\}}$$
$$squaring\ both\ sides$$
$$(x - 1)^2 + y^2 = x^2 + (y - 1)^2$$
$$x^2 - 2x + 1 + y^2 = x^2 + y^2 - 2y + 1$$
$$2y = 2x$$
$$y = x$$

Therefore the locus is a straight line $y = x$

Sub-topic 5: De Moivre's Theorem

Learning Outcomes

The learner should be able to:

i) prove De Moivres' theorem by mathematical induction.
ii) use De Moivres theorem to prove trigonometrical identities.
iii) simplify products and quotients of polar forms.
iv) find the roots of unity by using De Moivre's theorem and other complex numbers.

Lesson: De Moivre's Theorem

If n is an integer positive or negative or zero

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

Proof of the theorem may proceed as follows:

1. Prove that $(\cos\theta + i\sin\theta)(\cos\varphi + i\sin\varphi) = \cos(\theta+\varphi) + \sin(\theta_1+\varphi)$expand the left-hand side.
2. Use induction to prove the theorem for positive integral values of n.
3. Use the identity $(\cos\theta + i\sin\theta)\,(\cos\theta - i\sin\theta) = 1$ to show that
   a) $(\cos\theta + i\sin\theta)^{-1} = \cos n\theta - i\sin n\theta = \cos(-\theta) + i\sin(-\theta)$
   b) If $n = -m$, where m is a positive integer $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$. In accordance with the usual laws of algebra. $(\cos\theta + i\sin\theta)^0 = 1$.
4. If $n = \frac{1}{q}$ *where* q is an integer (positive or negative but not zero) show that one value of $(\cos\theta + i\sin\theta)^n$ is $\cos\frac{1}{q}\theta + i\sin\frac{1}{q}\theta$ by finding the value of $\left(\cos\frac{1}{q}\theta + i\sin\frac{1}{q}\theta\right)^q$
5. If n is a rational number say $\frac{p}{q}$ where p, q are integers, we have by stage 4 one value of
   $(\cos\theta + i\sin\theta)^{\frac{1}{q}} = \cos\frac{1}{q}\theta + i\sin\frac{1}{q}\theta$
   $(\cos\theta + i\sin\theta)^{\frac{p}{q}} = \left(\cos\frac{1}{q}\theta + i\sin\frac{1}{q}\theta\right)^p = \cos\frac{p}{q}\theta + i\sin\frac{p}{q}\theta.$

Question

a) $z = \cos\theta + i\sin\theta$. Show that $\frac{1}{1+z} = \frac{1}{2}(1 - i\tan\frac{\theta}{2})$
b) $z\bar{z} + 2iz = 12 + 6i. Find\ z$ (Crew equate real and imaginary)
c) $z_1 = 1 + i\sqrt{3}\,, z_2 = \sqrt{3} + i. Find\ the\ modulus\ and\ \arg of\ z_1z_2.$

If n is an integer positive or negative or zero

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

Prove by induction

Does it hold for $n = 1$?

LHS: $(\cos\theta + i\sin\theta)^1 = \cos\theta + i\sin\theta$

RHS: $\cos(1\times\theta) + i\sin(1\times\theta) = \cos\theta + i\sin\theta$

It Holds for $n = 1$

Does it hold for $n = 2$?

LHS: $(\cos\theta + i\sin\theta)^2 = \cos^2\theta + i(2\cos\theta\sin\theta) - \sin^2\theta = (\cos^2\theta - \sin^2\theta) + +i(2\cos\theta\sin\theta)$

$$= \cos 2\theta - i\sin 2\theta$$

RHS: $\cos(2\times\theta) + i\sin(2\times\theta) = \cos 2\theta + i\sin 2\theta$

It Holds for $n = 2.$

Assume it is holds for $n = k$.

$$(\cos\theta + i\sin\theta)^k = \cos k\theta + i\sin k\theta$$

Does it hold for $n = k + 1$?

LHS: $(\cos\theta + i\sin\theta)^{k+1} = (\cos\theta + i\sin\theta)^k(\cos\theta + i\sin\theta)^1 = (\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta)$

$$= \cos k\theta\cos\theta + i\cos k\theta\sin\theta + i\sin k\theta\cos\theta - \sin k\theta\sin\theta$$

$$= \cos k\theta\cos\theta - \sin k\theta\sin\theta + i(\cos k\theta\sin\theta + \sin k\theta\cos\theta)$$

$$= \cos(k\theta + \theta) + \sin(k\theta + \theta)$$

$$= \cos[(k+1)\theta] + \sin[(k+1)\theta]$$

RHS: $\cos\{(k+1)\theta\} + i\sin\{(k+1)\theta\}$

Since it holds for $n = k + 1$ *then* $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ *is true for all integral values of* n.

Simplify

2. $(\cos\theta + i\sin\theta)^5(\cos\theta - i\sin\theta)^3$

Using Demoivre's theorem

$$(\cos\theta + i\sin\theta)^5(\cos\theta - i\sin\theta)^3 = (\cos\theta + i\sin\theta)^5(\cos\theta + i\sin\theta)^{-3}$$

$$= (\cos\theta + i\sin\theta)^{5-3} = (\cos\theta + i\sin\theta)^2 = \mathbf{\cos 2\theta + i\sin 2\theta}$$

3. $\frac{(\cos\theta + i\sin\theta)^7}{(\cos\theta - i\sin\theta)^{-3}} = \mathbf{\cos 11\theta + i\sin 11\theta}$

$$\frac{(\cos\theta + i\sin\theta)^7}{(\cos\theta - i\sin\theta)^{-3}} = \frac{(\cos\theta + i\sin\theta)^7}{(\cos\theta + i\sin\theta)^{+3}} = (\cos\theta + i\sin\theta)^{7-3} = (\cos\theta + i\sin\theta)^4$$

Complete

1. $\sqrt{(\cos 2\theta + i\sin 2\theta)}$
2. $\sqrt[3]{(\cos 2\pi + i\sin 2\pi)}$
3. $\frac{1}{(\cos\theta + i\sin\theta)^2}$
4. $(\cos\theta - i\sin\theta)^{-3}$

Complex roots of unity

The equation of the n[th] degree has n roots. This means that the equation $z^3 - 1 = 0$ has three roots one of them is 1 but what are the others?

Using the identity $z^3 - 1 = (z-1)(z^2 + z + 1)$ to find the three cube roots of unity.

With de moivre's theorem. Now we can express 1 as a complex number in infinitely many ways: $\ldots \cos 2\pi + i\sin -2\pi$, $\cos 0 + i\sin 0$, $\cos 2\pi + i\sin 2\pi$, $\cos 4\pi + i\sin 4\pi$, … or in general $\cos 2k\pi + i\sin 2k\pi$where k is an integer.

Let z = r $(\cos\theta + i\sin\theta)$

In general $z = r(\cos(2k\pi + \theta) + i\sin(2k\pi + \theta))$ where k is a whole number.

$$z^{1/n} = r^{1/n}(\cos(2k\pi+\theta) + i\sin(2k\pi+\theta))^{1/n} = r^{1/n}\left(\cos\left(\frac{2k\pi+\theta}{n}\right) + i\sin\left(\frac{2k\pi+\theta}{n}\right)\right)^1$$

$$where\ k = 0,1,2,\ldots,n-1$$

Example

1. *Find the square root of 1*

   Let $z = 1 + 0i$

$$|z| = 1$$

   $\arg(z) = 0.$

$$z = \cos\theta + isin\,\theta$$

$$z = \cos(2\pi k + 0) + isin\,(2\pi k + 0)$$

$$z^{\frac{1}{2}} = \cos\left(\frac{2\pi k + 0}{2}\right) + isin\,\left(\frac{2\pi k + 0}{2}\right) = z = \cos\pi k + isin\,\pi k$$

   Where $k = 0,1$

$$when\ k = 0, z_1 = z^{\frac{1}{2}} = \cos 0 + isin\ 0 = 1$$

$$when\ k = 1, z_2 = z^{\frac{1}{2}} = \cos\pi + isin\,\pi = -i$$

2. *Cube root of unity*

   Let $z = 1 + 0i$

$$z = \cos\theta + isin\,\theta$$

$$z = \cos 2\pi k + isin\ 2\pi k$$

$$z^{\frac{1}{2}} = \cos\frac{2\pi k}{3} + isin\ \frac{2\pi k}{3}$$

$$where\ k = 0,1,2$$

$$when\ k = 0, z_1 = z^{\frac{1}{3}} = \cos 0 + isin\ 0 = 1$$

$$when\ k = 1, z_2 = z^{\frac{1}{3}} = \cos\frac{2\pi}{3} + isin\ \frac{2\pi}{3} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$$

$$when\ k = 2, z_3 = z^{\frac{1}{3}} = \cos\frac{4\pi}{3} + isin\ \frac{4\pi}{3} = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$$

Real and Imaginary Carts

$$If\ a+ib=c+id, where\ a,b,c,d\ are\ real\ roots. Then\ a=c\ and\ b=d.$$

Example

Prove that

1. $\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}$

By de moivre's theorem

$$\cos 3\theta + isin\ 3\theta = (\cos\theta + i\sin\theta)^3$$

The R.H.S may be written

$$(\cos\theta + i\sin\theta)^3 = \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta$$

$$\therefore \cos 3\theta + isin\ 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta + i(3\cos^2\theta\sin\theta - \sin^3\theta)$$

Equating real and imaginary parts

$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$$

$$sin\ 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$$

By division

$$\frac{sin\ 3\theta}{\cos 3\theta} = \frac{3\cos^2\theta\sin\theta - \sin^3\theta}{\cos^3\theta - 3\cos\theta\sin^2\theta}$$

Dividing numerator and denominator of the *R.H.S* by $\cos^3\theta$

$$\tan 3\theta = \frac{sin\ 3\theta}{\cos 3\theta} = \frac{\frac{3\cos^2\theta\sin\theta}{\cos^3\theta} - \frac{\sin^3\theta}{\cos^3\theta}}{\frac{\cos^3\theta}{\cos^3\theta} - \frac{3\cos\theta\sin^2\theta}{\cos^3\theta}} = \frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}$$

Therefore, $\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}$

2. $\tan 4\theta = \frac{4\tan\theta - 4\tan^3\theta}{1-6\tan^2\theta + \tan^4\theta}$

MATHEMATICS (MECHANICS)

SENIOR SIX MECHANICS

Topic: Moment of a Force

In this topic you will be able to:

i. find moment of a force.
ii. relate moments to real life experiences.
iii. identify clockwise, anticlockwise and zero moments.
iv. take moments about any given point.
v. distinguish between like and unlike parallel forces.
vi. use concept of parallel and nonparallel forces to find moment of a couple
vii. deduce that a system of forces forms a couple.
viii. determine the equation and position of the line of action of the resultant.

Materials

A rod (long stick), table, a pen, foolscaps or note books

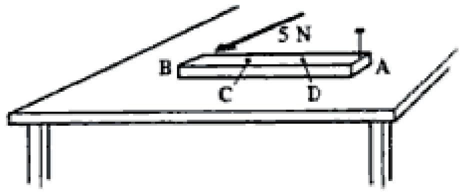
LESSON 1: To find moment of a force and relate it to real life experiences

Introduction:

In senior five you covered the topic on vectors as quantities with both magnitude and direction. We are now going to find the moment of a force and relate moment to real life experiences. When we find moments about a point, we also consider the sense of direction of the moment.

Instruction One:

Place the rod/stick on the table or ground. Fix one point of the rod with one hand and then push the rod at different points and in different directions.

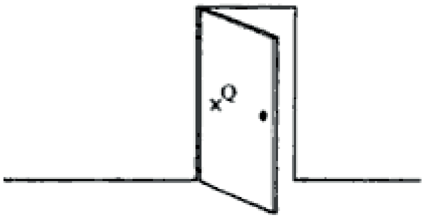


What do you observe as you move and turn the rod at different points with one end fixed?

What do you observe when you fix the rod at different points and apply the push at different points?

Instruction Two:

Open a window by pushing it at the farthest point from the hinges. Now push it at a point close to the hinges.



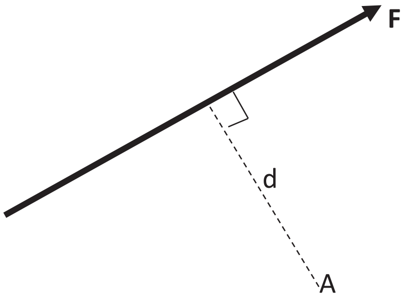
Compare the two turning effects. At which point was it easier to open the window?

Observation

When we apply force to any object which is partially fixed, there is always a turning effect. We refer to this turning effect as the moment of a force about a fixed point (as in instruction one) or about an axis (as in instruction two).

Finding moment of a force:

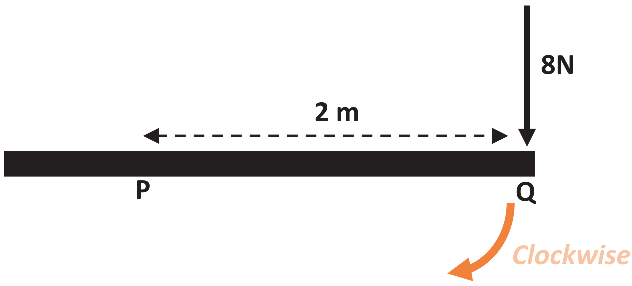
We find the **Moment of a force** about a point by multiplying the magnitude of the force by the perpendicular distance from the point to the line of action of the force. The moment of the force F (Newton) about the point A at a perpendicular distance d (metres) is F x d. Hence the unit of moments is Newton metres (Nm)



We also consider the direction of the turning effect (rotation). It should be either clockwise or anticlockwise. Moments of forces have both magnitude and direction.

Example One:

A force of 8N is applied at a point Q on the rod as shown in the diagram below. Find the moment of the force about the point P at a distance of 2m from Q.



Solution:

Moment of 8N about the point P = force x perpendicular distance from P

= 8 N x 2 m = 16 Nm clockwise

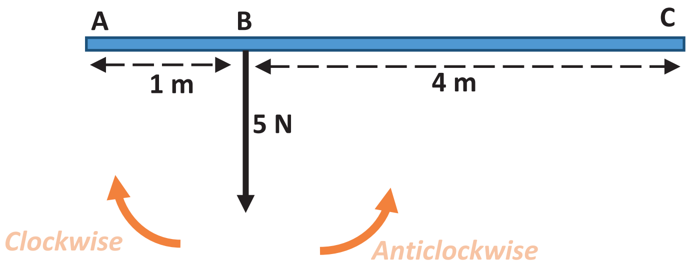
Observation:

i) The direction of the moment is always from the force direction towards the fixed point about which the moment is taken.

ii) Taking moments about a point, like A or about an axis through A may be donated as $\hat{A}$we read it as "moments about the point A"

Example Two:

In the diagram below, find the moment of the force of 5 N at B about each of the points A, B and C.



Solution:

Taking moments:

$\overset{\frown}{A}$ $5 \times 1 = 5\ Nm$ *Clockwise*

$\overset{\frown}{B}$ $5 \times 0 = 0\ Nm$ *Zero Moment*

$\overset{\frown}{C}$ $5 \times 4 = 20\ Nm$ *Anticlockwise*

Observation:

There is no turning effect about the point B since the force is acting at B. hence there is Zero moment.

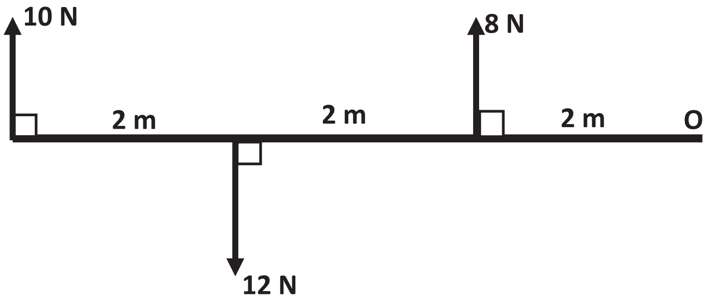
We cannot open the window if we push at the axis through the hinges. There would be no turning effect hence zero moment.

Sum of Moments:

We can have a number of forces acting on the same body and therefore we find the sum of the moments. However, the direction of each moment is considered.

Example Three

In the diagram below, forces 10 N, 12 N and 8 N act on a rod . Find the sum of the moments of the forces about the point O.



Solution:

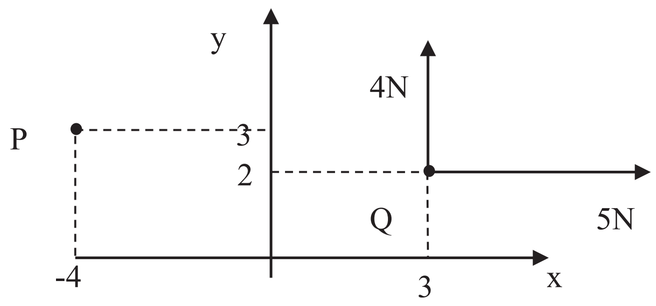
Sum of Moments about O = [(10 x 6) + (8 x 2)] *clockwise* + (12 x 4) *anticlockwise*

Taking anticlockwise as a negative moment, we shall have

= 60 + 12 – 48 = 24 Nm Clockwise

Example Four:

A force 5**i** + 4**j** acts at a point Q with position vector 3**i** + 2**j** . Find the moment of this force about the point P with position vector -4**i** + 3**j** .



Taking moments about P,

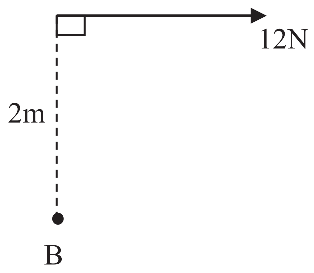
(5×7) anticlockwise + (4 × 1) anticlockwise

= 35 + 4 = 39Nm anticlockwise.

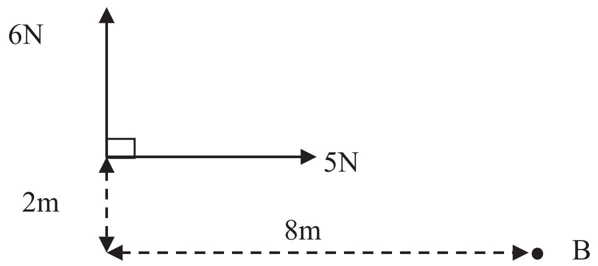
Activities

1. Four forces acting on a body have moments 9 Nm clockwise, 6 Nm anticlockwise, 18 Nm clockwise and 23 Nm anticlockwise about a point D. Find the sum of these moments in magnitude and direction.
2. Find the moments (or sum of the moments) about the point B of the forces shown in each of the diagrams below:

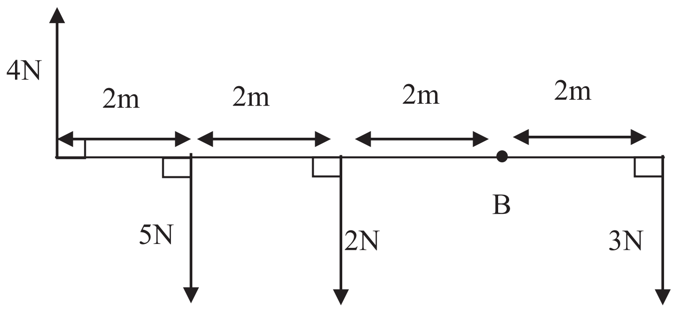
(a)



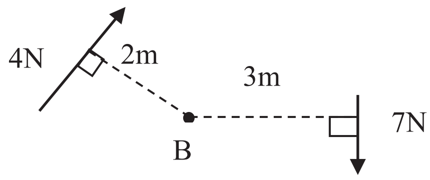
(b)



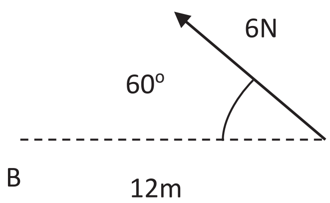
(c)



(d)



(e)



3. Find the moment of a force of 3**j** N acting at a point which has a position vector -5**i** m.
4. A force of (4**i** – 3**j**) N acts at the point which has a position vector (6**i** + 2**j**) m. Find the moment of this force about the point with the position vector (2**i** + 3**j**) m.
5. A force of (4**i**+ 2**j**) N acts at the point which has a position vector (4**i** + 4**j**) m and a force of 10**i** N acts at the point which has a position vector (-4**i** + 2**j**) m Find the sum of moments of these forces about the origin.

LESSON 2.To deduce that a system of forces forms a couple.

Introduction:

In the last lesson, we found the sum of moments of forces. We are now going to distinguish between like and unlike parallel forces, use the concept of parallel and nonparallel forces to find moment of a couple and deduce that a system of forces forms a couple.

Parallel Forces:

Like and unlike forces

We refer to forces which are parallel and act in the same direction as **like forces** while forces that are parallel but act in opposite directions, we refer to them as **unlike forces.**

A Couple

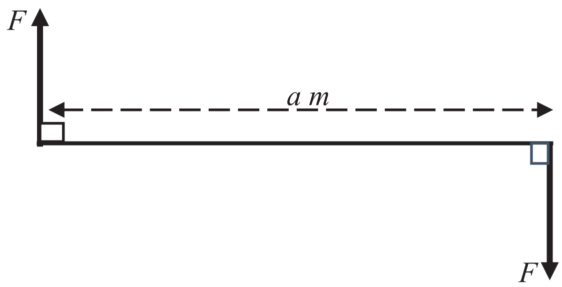
Two unlike forces of equal magnitude and not acting along the same line form a **Couple.** A couple has a **turning** effect but has no **translational** effect.

Observation:

When you apply equal forces at each end of a rod or stick, it will rotate but will not move from its position.

Moment of a Couple

If we denote the magnitude of each of the unlike forces in the couple as *F* Newtons and the perpendicular distance between the two forces as *a* metres, then we obtain the magnitude of the moment of the coupe as *F x a* Nm.

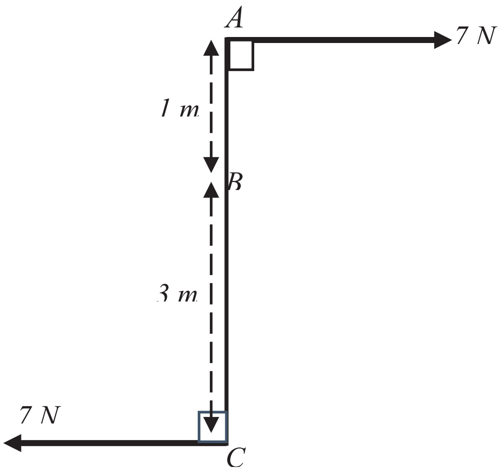


Observation:

The turning effect of a couple is **independent** of the point about which the turning is taking place. The moment of a couple is the **same** about all the points in the plane of the forces forming a couple.

Example One:

Find the sum of moments of the couple about each of the points A, B and C.



Solution:

Taking moments;

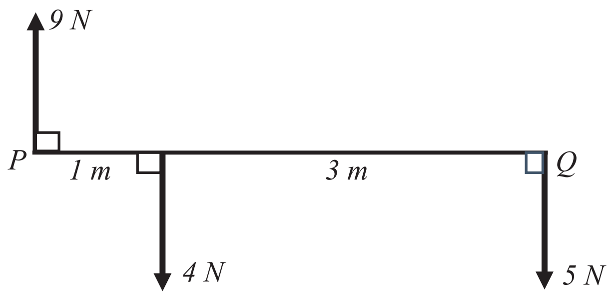
About A: Moments = (7 x 4) + (7 x 0) = 28 Nm *Clockwise*
About B: Moments = (7 x 3) + (7 x 1) = 28 Nm *Clockwise*
About C: Moments = (7 x 4) + (7 x 0) = 28 Nm *Clockwise*

Observation:

Three or more parallel forces may also form a couple.

Examples Two:

a) Show that the system of forces shown in the diagram below forms a couple.
b) Find the moment of the couple.



Solution:

a) Resolving the forces vertically, we have: 9 N – 4 N – 5 N = 0

   Observation: The forces have balanced in this direction and have no components hence there is no translation effect.

b) Taking moments about Q : (5 x 0) + (4 x 3) anticlockwise + (9 x 4) clockwise
=36 – 12 = 24 Nm clockwise.

Conclusion:

The system of forces balances in the vertical direction and has a turning effect of 24 Nm which shows that the forces reduce to form a couple.

Example Three:

Forces of 5N, 3N, 5N and 3N act along the sides QP, QR, SR and SP respectively, of a square PQRS, in the directions indicated by the order of the letters. The sides of the square is 4m.

(a) show that the forces form a couple.

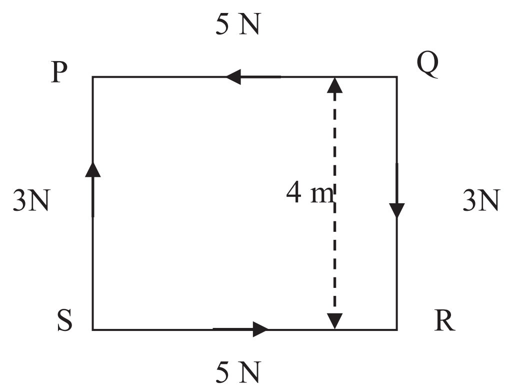
(b) Find the moment of this couple by taking moments about the:

(i) centre.

(ii) point P.

(c) Find the moment of the couple which must be applied to the system in order to produce equilibrium.

(a)



Resolving parallel to PQ -5 + 5 = 0N

Resolving parallel to PS -3 + 3 = 0N

Hence, the system is not translated. So, it is either in equilibrium or the forces reduce to equilibrium.

(b) (i) Taking moments about the centre. (3 × 2) - (5 × 2) + (3 × 2) - (5 × 2) = 6- 10 + 6 – 10
= - 8Nm (8Nm anti clockwise)

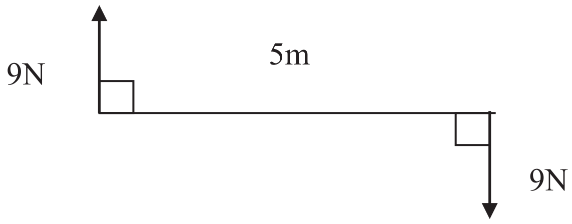
(i) Taking moments about P. - (5 × 4) + (3 × 4) = - 20 + 12
= - 8Nm (8Nm anti clockwise)

(c) 8Nm clockwise.

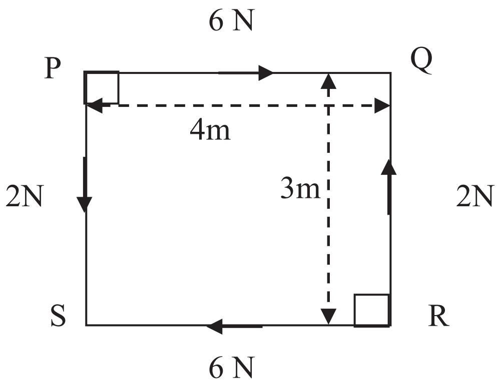
ACTIVITIES:

1. Which of the systems below will reduce to a couple? For each, find the moment of the couple.

(a)



(b)



2. Forces of 8N, 5N, 8N and 5N act along the sides BA, BC, DC and DA respectively, of a rectangle ABCD, in the directions indicated by the order of the letters. If AB is 2m and BC is 5m

(a) show that the forces form a couple.

(b) Find the moment of this couple by taking moments about the:

(i) centre.

(ii) point P.

3. A force of $(4\mathbf{i}+ 3\mathbf{j})$ N acts at the point which has a position vector $(6\mathbf{i} + 3\mathbf{j})$ m and a force of $(-4\mathbf{i} - 3\mathbf{j})$ N acts at the point which has a position vector $(3\mathbf{i} - \mathbf{j})$ m. Show that the system reduces to a couple and find the moment of the couple.

LESSON 3

Introduction:

In the last lesson we found the moment of a couple and the conditions for a system of forces to form a couple. In this lesson we are going to find the resultant of forces. We shall also determine the position of the line of action of the resultant.

Resultant Force To use principle of moments to find resultant of a system of forces.

When a system of forces does not reduce to a couple, then these forces in the system have a resultant force with both a translational effect and turning effect.

The magnitude of the resultant force is obtained by resolving the forces in the system (translational effect). The turning effect of the resultant is equivalent to the sum of moments of the forces.

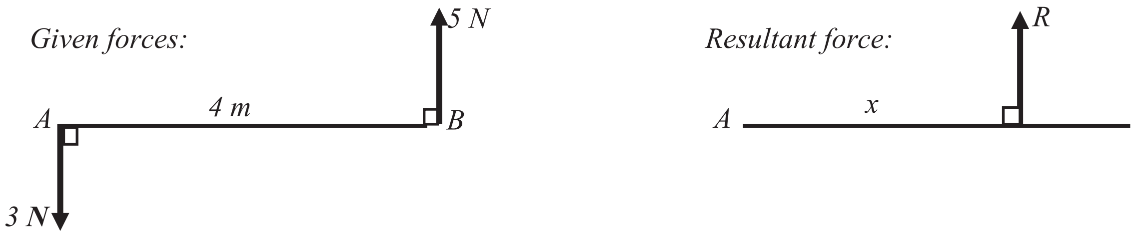
Principle of Moments:

When a number of coplanar forces act on a body, the algebraic sum of the moments of these forces about any point in their plane, is equal to the moment of the resultant of these forces about that point.

Example One:

Two unlike parallel forces of 3 N and 5 N are 4 m apart. Find the direction, magnitude and line of action of the resultant of these forces.

Solution:



We first draw two diagrams, one for the forces and another for the resultant force.

Let x be the distance of R from the line of action of 3 N force at A.

Resolving forces vertically, we have; R = 5 – 3 = 2 N.

Taking moments about A: (3 x 0) + (5 x 4) = R x x *(Principle of moments)*

$$R \times x = 20 \text{ Nm}$$

$$2 \text{ x } x = 20 \text{ Nm}$$

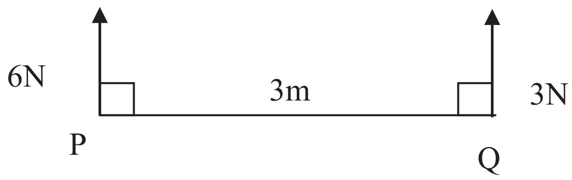
$$x = 10 \text{ m}$$

The resultant acts at a distance of 10 m from A, has a magnitude of 2 N and acts in the same direction as the 5 N force.

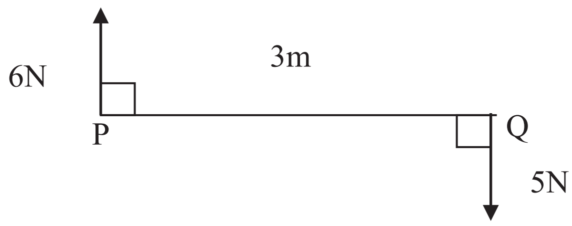
ACTIVITIES

1. In each of the following cases, find the magnitude of the resultant of the forces shown and the distance of its line of action from P.

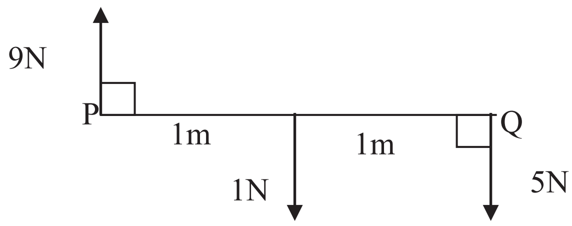
(a)



(b)



(c)



2.The lines of action of two unlike forces of 4N and 6N are 5m apart. Find the resultant of the forces and the distabce between its kine of action and that of the 4N force.

TOPIC 44: COPLANAR FORCES IN EQUILIBRIUM

In this topic, you will be able to:

- Apply the conditions for forces in equilibrium to exist.
- Apply the principle of moments to solve problems on:
  - Ladders
  - Rods

Materials: A rod/stick, strings/thread/sisal/fibre, weights/stones, block, pen, pencil, ruler.

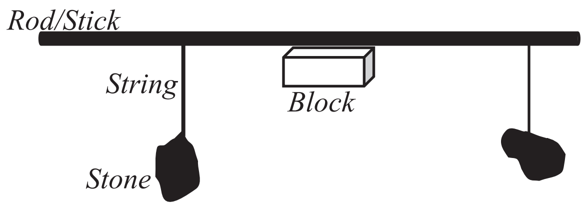
LESSON 1: To apply the conditions for forces in equilibrium to exist

Introduction:

In the last topic, we found moments of forces, a couple and also a resultant force for a system of forces. We are now going to find out the conditions for forces to be in equilibrium and apply the principle of moments.

Instruction One:

Tie one end of a string around a stone and the other end to the rod/stick. Tie one end of another string around another stone and the other end to the rod/stick. Place the rod/stick with the stones hanging on the block. Keep adjusting the position of the strings on the rod/stick until it balances as shown below.



Observation:

i) There is no translation effect as the stones are balancing. (Resultant force is zero)
ii) There is no turning/rotational effect as the rod/stick is horizontal (moments are zero)

Conditions for forces in equilibrium:

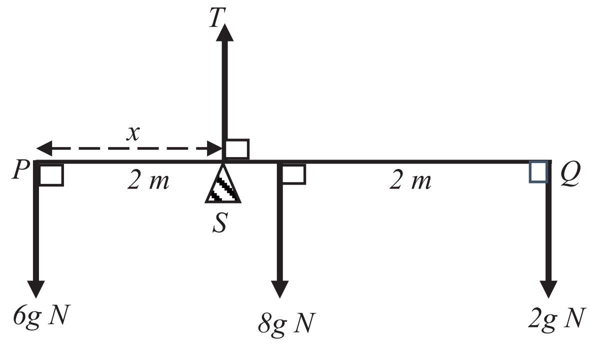
i) The resultant force in any direction is zero.
ii) The algebraic sum of moments of the forces about any point is zero. (clockwise and anticlockwise moments are equal)

Example:

A uniform rod of length 4 m and mass 8 kg, has a mass of 6 kg attached at one end and a mass of 2 kg attached at the other end. Find the position of the support if the rod rests in a horizontal position.

Solution:

For uniform rods, the weight is in the middle of the rod. Hence 8g N is 2 m from one end of the 4 m rod.



Equating the clockwise and anticlockwise moments about the Support S:

$$(6g \times x) = (8g \times (2 - x)) + (2g \times (4 - x))$$

$$6g\,x = 16g - 8gx + 8g - 2g\,x$$

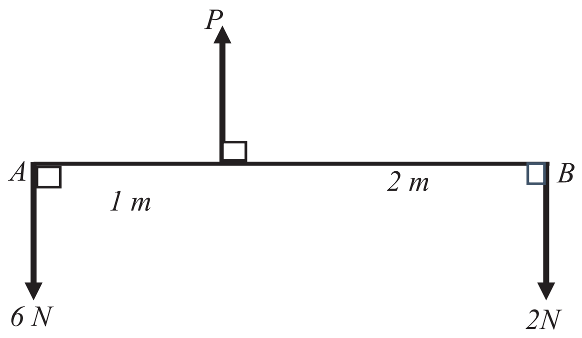
$$16g\,x = 24g$$

$$x = 1.5 \text{ m}$$

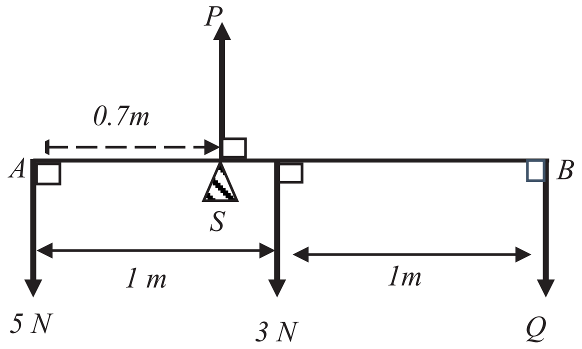
ACTIVITIES

1. Each of the figures below show light horizontal rods AB in equilibrium under the action of the forces indicated, For each case, find the values of P and Q where applicable.

(a)



(b)



2. A uniform beam PQ of mass 6kg and length 4m rests horizontally supported at its ends. A mass of 10kg hangs from the beam at a distance of 1m from A. Find the reactions at the supports.

LESSON 2: To apply the principle of moments to solve problems on:
- Ladders
- Rods

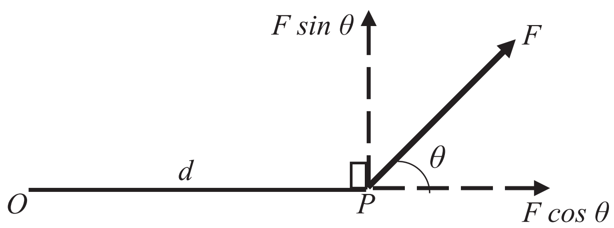
Introduction:

In the last lesson, we applied conditions for forces in equilibrium and the principle of moments to parallel forces. We are now going to apply the principle of moments to solve other problems with non-parallel forces.

Non-Parallel forces in equilibrium:

We also apply the same conditions of forces in equilibrium to non-parallel forces. When we are finding the moments of a non-parallel force about a point, we shall resolve the force into components.

Force F acts at a point P and at a distance of d from the point O. F makes an angle of θ with OP produced as shown below.



Resolving the force F we have,

Component in the direction *OP* is $F \cos \theta$
Component at right angles to *OP* is $F \sin \theta$

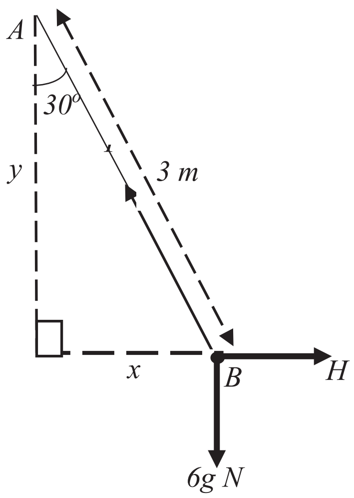
Moment of the force F about O = sum of moments of the components

$$= F \cos \theta \times 0 + F \sin \theta \times d$$

$$= Fd \sin \theta$$

Example One:

A pendulum AB consists of a string, of length 3 m and a bob of mass 6 kg. The pendulum is suspended from A and held in equilibrium by a horizontal force H applied at B so that the string makes an angle of $30^o$with the vertical. Find the force H and the Tension in the string.



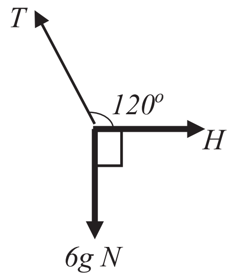
Taking moments about A: $6g \times x = H \times y$

$6g \times 3\sin 30^o = H \times 3 \cos 30^o$

$6g \times 3 \times ½ = H \times 3 \times \sqrt{3}/2$

$H = 6g/\sqrt{3} = 33.95$ N

To find the Tension in the string, we apply Lami's theorem of three forces in equilibrium:

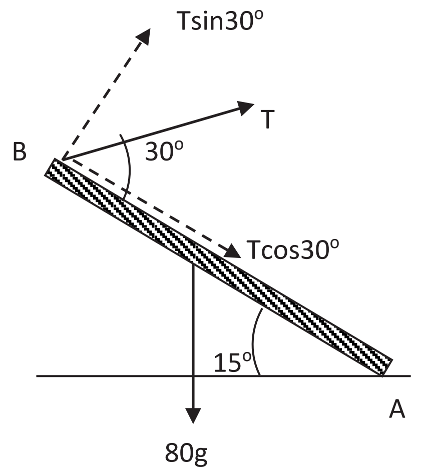


$$\frac{T}{\sin 90^\circ} = \frac{6g}{\sin 120^\circ} = \frac{H}{\sin 150^\circ}$$

$$T = \frac{6g \times \sin 90°}{\sin 120°} = 67.9\ N$$

Example Two:

A uniform beam AB of mass 80 kg and length 2l m has its lower end A resting on a rough horizontal ground. It is held in equilibrium at an angle of 15°with the horizontal by a rope attached to the end B which makes an angle of 30° with BA. Find the tension in the rope.



Taking moments about A: (Any force at A has zero moment)

$$80g \times l \cos 15° = (T \sin 30° \times 2l) + (T \sin 30° \times 0)$$

$$T = 758.06\ N$$

ACTIVITY.

1. A uniform beam AB of length 2m and weight 5N is hinged at A to a vertical wall. It is supported horizontally by a string at B inclined at 50° to the line AB. Find the:

(a) tension in the string.

(b) reaction at the hinge.

2. A uniform beam AB of mass 6 kg and length 2l m has its lower end A resting on a rough horizontal ground. It is held in equilibrium at an angle of 15°with the horizontal by a rope attached to the end B which makes an angle of 30° with BA. Find the

(a) tension in the rope.

(b) reaction at A.

3. A pendulum AB consists of a string, of length 4 m and a bob of mass 8 kg. The pendulum is suspended from A and held in equilibrium by a horizontal force F applied at B so that the string makes an angle of $20^{o}$with the vertical. Find the force F and the Tension in the string.

TERM 2

Topic: CIRCULAR MOTION

In this topic you will be able to:

- derive the relationship between linear and angular speed.
- relate the force acting towards the centre of the circle and the acceleration, when an object is in circular motion.
- apply the relationship to motion of a particle on a string.

Materials:

a string/sisal, a bob/stone, a stick, a pen, foolscaps or note books,

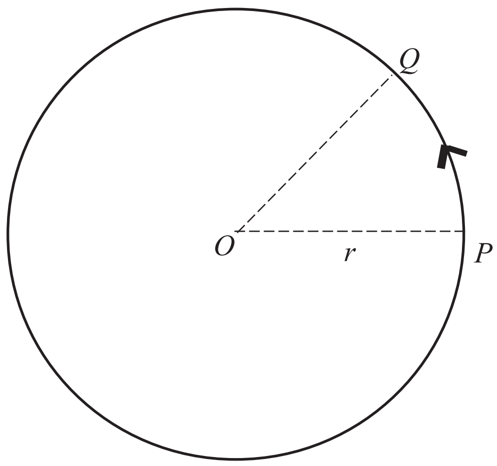
LESSON 1 To derive the relationship between linear and angular speed.

Introduction:

In senior five you covered the topic on linear motion. We are now going to derive the relationship between linear and angular speed. Angular speed is the rate of change of the angle at the centre of a circle, along which an object is moving. We shall also relate the force acting towards the centre of the circle and the acceleration, when an object is in horizontal circular motion.

Instruction:

Let us consider a body moving in a circular motion with a linear speed v ms^{-1} and angular speed ω rad s^{-1} (radians per second). The radius of the circle being r m and centre O.



In one second: distance PQ = v m and also the angle POQ = ω *radians.*

We shall now use similarity: $\frac{PQ}{2\pi r} = \frac{\omega}{2\pi} \rightarrow \frac{v}{r} = \omega \rightarrow \therefore v = r\omega$

Linear speed = radius of the circle x angular speed.

When we, we shall have: $acceleration = \frac{v^2}{r}$

Central Force:

A body moving in a circular motion has acceleration directed towards the centre of the circle hence there is also a force acting towards the centre.

We apply Force = mass x acceleration; then the $Central\ Force = m\frac{v^2}{r}$

Example One:

A body of mass 500g moves with constant angular speed of 8 rad s^{-1} in a horizontal circle of radius 5m. Calculate the force that must act on the body towards the centre of the circle.

Solution:

$$Central\ Force = m\frac{v^2}{r} \quad but \quad v = r\omega$$

$$Central\ Force = m\frac{r^2\omega^2}{r} = mr\omega = \frac{500}{1000} \times 5 \times 8 = 10\ N$$

ACTIVITIES

1. A particle moves with a constant angular speed of 4 rad/s around a circular path of radius 3 m. Calculate the acceleration of the particle.
2. A body of mass 5 kg moves with a constant angular speed of 3 rad/s around a horizontal circle radius 8 cm. Find the magnitude of the horizontal force that must be acting on the body towards the centre of the circle.
3. A body of mass 2 kg moves with a constant speed of 5 m/s around a horizontal circle of radius 2 m. Find the magnitude of the horizontal force that must be acting on the body towards the centre of the circle.

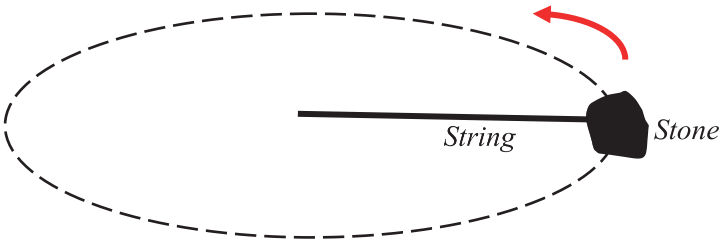
LESSON 2 To relate the force acting towards the centre of the circle and the acceleration, when an object is in circular motion.

Introduction:

In last lesson, we derived the relationship between linear and angular speed. We also related the force acting towards the centre of the circle and the acceleration. We are now going to apply the relationship to motion of a particle on a string. We shall also consider the tension in the string when a body performs circular motion.

Instruction One:

Tie one end of the string to the bob/stone. Fix the other end of the string to the ground or on the table. Pull the stone until the string is taut and push it. What do you observe?



Observation:

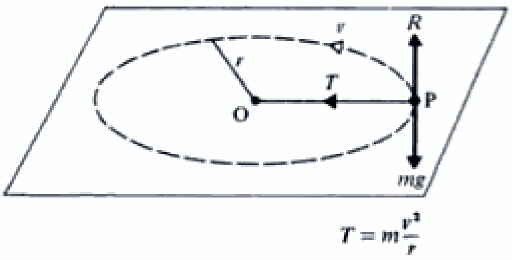
The stone performs a circular motion only if the string is taut hence the central force keeping the stone in the horizontal circular motion is the tension in the string.

Example One:

A particle of mass 200gis attached to one end of a light inextensible string of length 30cm. The other of the string is fixed at O on a smooth horizontal surface. The particle describes a circle with Centre O. Find the tension in the string when the speed of the particle is 4 ms^{-1}

Solution:

The acceleration is towards the centre O hence the central force is also towards the centre and this is the tension force.

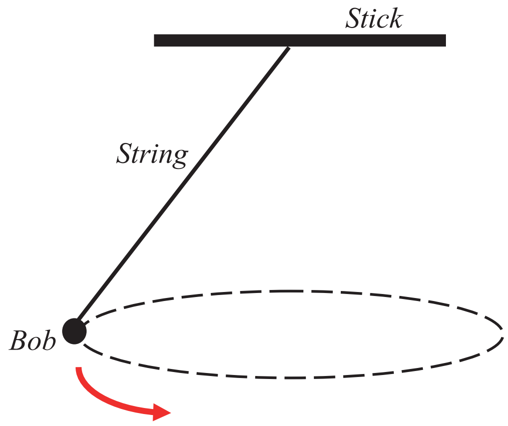


$$Tension = Central\ Force = m\frac{v^2}{r} = \frac{200}{1000} \times \frac{4^2}{0.3} = 10.7N$$

Conical Pendulum

Instruction Two:

Tie one end of a string to a stone/bob. Fix the other end of the string vertically up on a horizontal fixed stick. Then push the bob/stone to describe a horizontal circle.

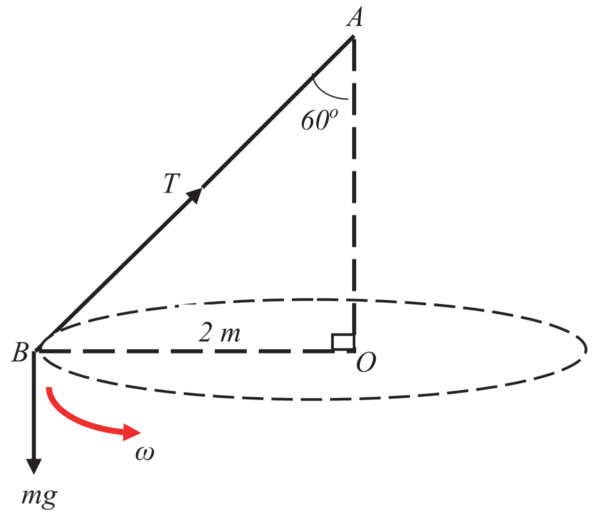


As the bob describes a horizontal circle, the string also describes a curved surface of a cone hence this system is referred to as a **conical pendulum**.

Example Two:

A conical pendulum consists of a light inextensible string AB, fixed at A and carrying a mass a particle of 100g at B. The particle moves in a horizontal circle of radius 2 m and centre vertically below A. If the angle between the string and the vertical is 60º, determine the tension in the string and the angular speed of the particle.

Solution:



Resolving vertically: $T \cos \theta = mg$

$T \cos 60^o = 0.1 \times 9.8$

$T = 1.96$ N

Horizontal component of the tension is the central force.

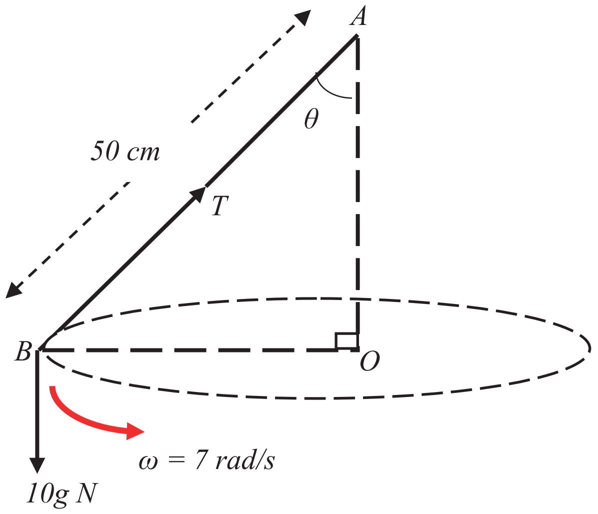
$T \sin \theta = mr\omega^2$

$1.96 \sin 60^o = 0.1 \times 2 \times \omega^2$

Angular speed $\omega = 2.91$ rad s^{-2}

ACTIVITIES:

1. A particle of mass 500g lies on a smooth horizontal surface and is connected to a point O on the surface by a light inextensible string of length 30 cm. With the string taut, the particle describes a horizontal circle, centre O with a constant angular speed of 30 rad/s. Determine the tension in the string.
2. A body of mass 5 kg lies on a smooth horizontal surface and is connected to a point O by a light inextensible string of length 2 m. With the string taut, the body describes a horizontal circle with centre O. If the tension in the string is 40 N, find the speed of the body.
3. The diagram below shows a conical pendulum consisting of a light inextensible string AB fixed at A and carrying a bob of mass 10 kg at B. The bob moves in a horizontal circle with the centre vertically below A and a constant angular speed of 7 rad/s. Find the values of T the tension and θ.



4. A conical pendulum consists of a light inextensible string PQ of length 40 cm fixed at P and carrying a bob of mass3 kg at Q. The bob describes a horizontal circle about the vertical through P with a constant angular speed of 4 rad/s. Calculate the tension in the string.

Topic: ELASTICITY

In this topic you will be able to:

i) distinguish between natural length, extension and compression.
ii) State and use Hooke's law.
iii) use modulus of elasticity.
iv) Solve problems involving one elastic string or spring with a mass attached at one end.
v) Calculate elastic potential energy stored in a string or spring.

Materials

an elastic string/spring/rubber band, a bob/stone, a stick, a pen, foolscaps/note books,

LESSON 1Relating Hookes law to elastic springs.

Introduction:

In the last topics, we considered situations of equilibrium or motion when particles/bodies are attached to inelastic strings. We are now going to consider situations when the strings are **elastic**. An elastic string has its **natural length** before it is stretched by an external force to a new length. The difference between the natural length and the new length is the **extension**. When the force is removed, the string returns to its original length which is the natural length. We also have elastic springs that are either stretched or compressed.

Elastic Strings

Instruction One:

a) Measure the natural length l of any rubber band.
b) Stretch the rubber band/spring and measure the new length $l'$of the rubber band.
c) Calculate the difference between the two lengths and this gives the extension x caused by the force you exerted as you stretched the rubber band.
d) When you let go of the rubber band, it will return to its original natural length.

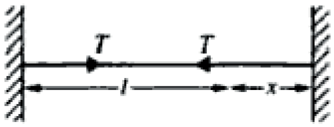
Observation: **Extension x = *Stretched length, l'* – *Natural length, l***

Hooke's Law:

This law states that the tension in the string is proportional to the extension of the string from its natural length.

Tension, T α Extension, x

$$T = \lambda \frac{x}{l}$$



Where T is the tension in the string, x is the extension, l is the natural length, λ is the modulus of elasticity of the string whose units are newtons.

Example One:

An elastic string is of natural length 4 m and modulus of elasticity of 20 N. Find:

a) The tension in the string when the extension is 50 cm.
b) The extension of the string when the tension is 4 N.

Solution:

a) Applying Hooke's law $\boldsymbol{T} = \lambda\frac{x}{l} = 20 \times \frac{0.5}{4} = 2.5\ N$ ∴*The tension is 2.5 N.*

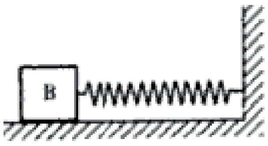
b) $\boldsymbol{T} = \lambda\frac{x}{l} \rightarrow x = \frac{T \times l}{\lambda} = \frac{6 \times 4}{20} = 1.2\ m$ ∴*The extension is 1.2 m.*

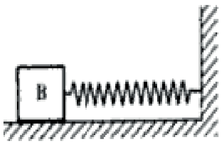
Elastic Springs

A spring is elastic and it stretches. However, unlike an elastic string, the elastic spring also compresses when a force is applied and its natural length is decreased. The force in the spring as it is compressed is now referred to as a ***Thrust*** and not the *Tension.*

We apply Hooke's law to both a stretched spring and a compressed spring.

Natural Length ***Stretched Spring*** ***Compressed Spring***





Example Two:

A spring is of natural length 2.5 m and modulus of 35 N. Calculate the thrust in the spring when it is compressed to a length of 2.2 m.

Solution:

Compression = natural length – compressed length = 2.5 – 2.2 = 0.3 m.
Applying Hooke's law $Thrust\ \boldsymbol{T} = \lambda\frac{x}{l} = 35 \times \frac{0.3}{2.5} = 4.2\ N$ ∴*The thrust is 4.2 N.*

ACTIVITIES

1. A spring is of natural length 40 cm and modulus 15 N. Find the thrust in the spring when it is compressed to a length of 30 cm.
2. When the tension in a spring is 5 N, the length of the spring is 15 cm longer than its natural length of 65 cm. Find the modulus of the spring.
3. Find the extension of an elastic string of natural length 70 cm and modulus 16 N when the tension in the string is 6 N.
4. When an elastic string is stretched to a length of 20 cm the tension in the string 16 N. If the modulus of the string is 30 N, find the natural length of the string.

LESSON 2: Relating Hookes law to Equilibrium of a suspended body

Introduction:

In the last lesson, we applied Hooke's law to find the tension and thrust in strings and springs. We are now to going to consider particles attached to strings which are elastic or attached to elastic springs.

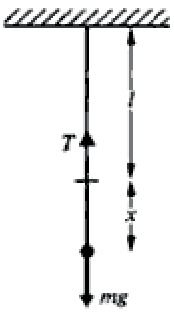
Equilibrium of a suspended body

Instruction:

a) Tie one end of an elastic end to a fixed point.
b) Measure the natural length of the string.
c) Attach a stone to the other end of the string and let the system hang freely vertically down. What do you observe?
d) Measure the stretched length of the string.

Observation

The string is stretched due to the force caused by the weight of the stone.
The system is in equilibrium since there is no translatory nor rotational effect.



Resolving vertically: T = mg

Applying Hooke's law $\boldsymbol{T} = \lambda\frac{x}{l} \rightarrow mg = \lambda\frac{x}{l}$

Example One:

A light elastic string of natural length of 65 cm has one end fixed and a mass of 600g freely suspended from the other end. Find the modulus of the string if the total length of the string in equilibrium position is 85 cm.

Solution:

Extension $x = 85 - 65 = 20$ cm $= 0.2$ m.

Resolving vertically and applying Hooke's law we have

$$mg = \lambda \frac{x}{l} \to \lambda = \frac{mgl}{x} = \frac{0.6 \times 9.8 \times 0.65}{0.3} = 19.1\ N\ (Modulus)$$

Example Two:

A light elastic spring has its upper end P fixed and a body of mass 0.5 kg attached to its other end Q. If the modulus of the spring is 2.5g N and its natural length 1.2 m. Find the extension of the spring when the body hangs in equilibrium.

Solution:

Resolving vertically and applying Hooke's law we have

$$mg = \lambda \frac{x}{l} \to x = \frac{mgl}{\lambda} = \frac{0.5 \times g \times 1.2}{2.5g} = 0.24\ m\ Extension\ is\ 24\ cm$$

ACTIVITIES

1. A light elastic string of natural length 30 cm and modulus 3g N has one end fixed and a mass of 600g freely suspended from the other end. Find the total length of the string.
2. A light spring of natural length 50 cm and modulus 2g N has one end fixed and a body of mass 1.5 kg freely suspended from its other end.
   a) Find the extension of the spring.
   b) What mass would cause the same length of extension when suspended from a spring of natural length 40 cm and modulus 1.2g N?
3. When a mass of 5 kg is freely suspended from one end of a light elastic string the other end of which is fixed, the string extends twice its natural length. Find the modulus of the string.

LESSON 3: Calculating Potential Energy stored in a string.

Introduction:

In the last lesson, we considered particles attached to strings which are elastic or attached to elastic springs and freely hanging vertically in equilibrium. We are now going to calculate elastic potential energy stored in a string or spring.

Potential Energy stored in a string

When we stretch a string, there is work done and energy is stored which is Potential Energy (PE). The string then returns to its natural length using the stored energy which converts to Kinetic Energy (KE) during the movement back. This brings us to the principle of conservation of energy.

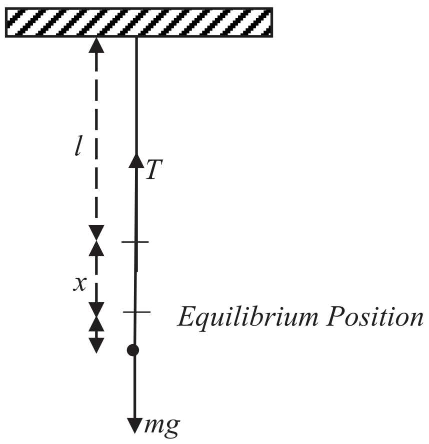
Total Energy = PE due to gravity + PE stored in the string + KE = Constant
Initial Total Energy = Final Total Energy
Energy stored in the string due to stretching through an extension x $= \frac{\lambda x^2}{2l}$

Instruction:

Follow the steps a) to d) in the instruction given in the last lesson two.

e) Pull the stone further down and measure the new length.
f) Release the stone. What do you notice?

Observation



The string gained potential energy due to gravity when we attached the stone hence the first extension *x*. When we pull down the stone then release it, it moves back to its *equilibrium position* where it was hanging freely. The kinetic energy at a certain velocity causes the movement back to the equilibrium position.

Example One:

An elastic string is of natural length 1.6 m and modulus 25 N. Find the energy stored in the string when it is extended to 1.9 m.

Solution:

Extension $x = 1.9 - 1.6 = 0.3$ m.

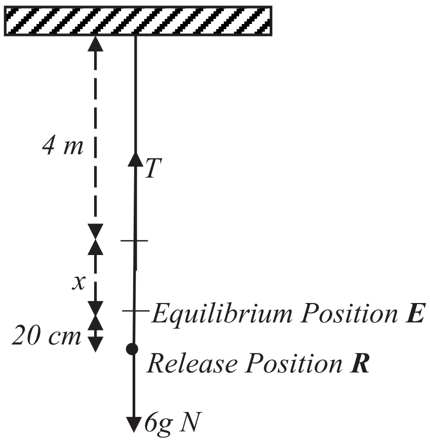
$$Energy\ stored = \frac{\lambda x^2}{2l} = \frac{25 \times (0.3)^2}{2 \times 1.6} = 0.7\ J$$

Example Two:

A light elastic string is of natural length 4 m and modulus 30g N. One end of the string is attached to a fixed point and a body of mass 6 kg hangs in equilibrium from the end. The body is pulled down 20 cm and then released. Determine:

a) The extension of the string in equilibrium position.
b) The energy stored in the string just before the body is released.
c) The speed of the body as it passes through the equilibrium position.

Solution:



a) The extension x of the string in equilibrium position.

Resolving vertically and applying Hooke's law we have

$$mg = \lambda \frac{x}{l} \rightarrow x = \frac{mgl}{\lambda} = \frac{6 \times g \times 4}{30g} = 0.8\ m\ Extension\ is\ 0.8\ m$$

b) The energy stored in the string ***R,*** at just before the body is released.

$$KE = \frac{1}{2}mv^2 = \frac{1}{2} \times 6 \times 0^2 = 0\,J$$

$$PE\ due\ to\ gravity\ = mgh = 6 \times g \times (-0.2) = -1.2g\,J$$

$$PE\ in\ string\ = \frac{\lambda b^2}{2l} = \frac{30g\ \times (0.8 + 0.3)^2}{2 \times 4} = 3.75g\,J$$

$$Total\ Energy\ at\ \boldsymbol{R} = -1.2g\ + 3.75g = 2.55g\,J$$

c) The speed v ms^{-1}of the body as it passes through the equilibrium position ***E***.

$$KE = \frac{1}{2}mv^2 = \frac{1}{2} \times 6 \times v^2 = 3v^2\,J$$

$$PE\ due\ to\ gravity\ = mgh = 6 \times g \times (0) = 0\,J$$

$$PE\ in\ string\ = \frac{\lambda x^2}{2l} = \frac{30g\ \times (0.8)^2}{2 \times 4} = 2.4g\,J$$

$$Total\ Energy\ at\ \boldsymbol{E} = 3v^2\ + 2.4g$$

We now apply the principle of conservation of energy,

Total energy at ***R*** = Total energy at ***E***

$$2.55g = 3v^2\ + 2.4g \rightarrow 3v^2 = 0.15g \rightarrow \quad \therefore v = 0.7\ ms^{-1}$$

Observation

The potential energy stored in springs has the same expression either when it is stretched or when it is compressed.

ACTIVITIES

1. Find the work that must be done to stretch an elastic string of modulus 300 N from its natural length of 3 m to a stretched length of 3.5m.
2. Find the work that must be done to compress a spring of modulus 400 N from its natural length of 9 cm to a shortened length of 7 cm.
3. A light elastic string is of natural length of 60 cm and modulus 150 N. One end of the string is attached to a fixed point and a body of mass 4 kg is freely suspended from the other end. Find
   a) The extension of the string in the equilibrium position
   b) The energy stored in the string
4. A body of mass 4 kg is freely suspended from a spring of natural length 85 cm and modulus of 7g N. The other end of the spring is fixed to a point A. The body initially hangs freely in equilibrium at a point B. It is then pulled down a further distance of 35 cm to a point C and released from rest. Find
   a) The distance AB
   b) The energy stored in the spring when the body rests at B
   c) The energy stored in the spring when the body is held at C.
   d) The kinetic energy of the body when it passes through B after release from C.

TOPIC: **SIMPLE HARMONIC MOTION**

In this topic you should be able to:

i) identify amplitude, period, displacement and angular velocity.
ii) determine expressions relating to S.H.M
iii) calculate maximum velocity and find the acceleration.
iv) aerivation of the equation $x + \omega^2 x = 0$.
v) apply the expressions of S.H.M for horizontal and vertical springs.

Materials:

an elastic string/spring/rubber band, a bob/stone, a stick, a pen, foolscaps/note books

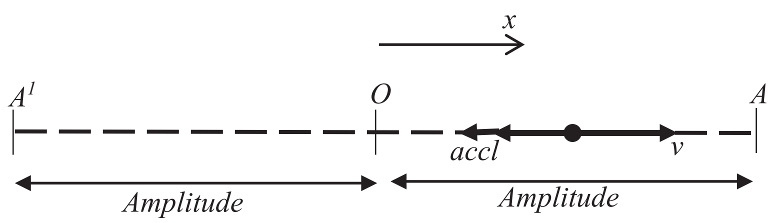
LESSON 1: Identifying and using the equations of simple harmonic motion

Introduction: .

In the last topic of elasticity, we considered particles attached to strings which are elastic or attached to elastic springs and freely hanging vertically in equilibrium. We also solved problems concerning elastic potential energy stored in a string or spring. We are now going to consider particles performing Simple Harmonic motion (S.H.M).

Instruction One:

When a particle moves in a straight line through a fixed point O and its acceleration is directed through O and its displacement from O is always proportional to the acceleration then the particle has performed ***Simple Harmonic Motion (S.H.M).***



Observation

- The particle oscillates between A^1 through O to A and back.
- $OA = OA^1$ and is called the ***Amplitude***.
- Velocity at A and A^1 is always zero as the particle momentarily stops and returns.
- Movement from A to A^1 and back to A is ***one cycle***.

Since acceleration α *x,* we now use n^2 as the constant of proportionality and we obtain;

$$\ddot{x} = -n^2 x$$

Using acceleration as $v\frac{dv}{dx}$, we obtain:

$$v\frac{dv}{dx} = -n^2 x$$

$$\int v dv = \int -n^2 x\, dx$$

$$\frac{v^2}{2} = \frac{-n^2x^2}{2} + C$$

Since $v = 0$ when $x = \pm a$ then:

$$\frac{0}{2} = \frac{-n^2x^2}{2} + C \rightarrow C = \frac{n^2a^2}{2}$$

$$v^2 = n^2(a^2 - x^2)$$

We also note that, maximum velocity is when the particle is at *O* and $x = 0$ hence we obtain:

$$v_{max} = na$$

Other equations important equations applied in S.H.M are:

$$x = a\sin(nt + \epsilon) \text{ where } \epsilon \text{ is the constantof integration}$$
$$v = an\cos(nt + \epsilon)$$
$$T = \frac{2\pi}{n} \quad \text{where } T \text{ is the period, time for one complete cycle.}$$

Example One:

A particle is moving with SHM of period $\frac{\pi}{5}$ s and amplitude 36 cm. Find the maximum speed and the maximum acceleration of the particle.

Solution:

$$T = \frac{2\pi}{n} = \frac{\pi}{5} \quad \rightarrow \quad n = 10$$
$$v_{max} = na = 10 \times 0.36 = 3.6\, ms^{-1}$$

Acceleration is maximum when $x = -a$

$$\ddot{x} = -n^2x \rightarrow \ddot{x}_{max} = -(10)^2(-0.36) = 36\, ms^{-2}$$

Example Two:

A particle moves with SHM about a mean position O. When the particle is 40 cm from O, its speed is 2.5 ms^{-1} and when it is 110 cm from O, its speed is 1.0 ms^{-1}. Find the amplitude and periodic time of the motion.

$2.5^2 = n^2(a^2 - 0.4^2)$

$1.0^2 = n^2(a^2 - 1.1^2)$

Dividing the two equations: $\frac{2.5^2 = n^2(a^2-0.4^2)}{1.0^2 = n^2(a^2-1.1^2)} \rightarrow \quad a = 1.2\ m \ \ (amplitude)$

Substituting for *a*: $1.0^2 = n^2(1.2^2 - 1.1^2) \rightarrow \quad n = 2.09 \quad \therefore T = \frac{2\pi}{2.09} s \quad (Period)$

Circular motion and S.H.M

A particle of mass *m* moving in a circle with angular velocity *ω*, the radius of the circle being, if P is executing S.H.M with constant velocity *v* ms^{-1}, then the equations we use will have $n = \omega$:

$\ddot{x} = -\omega^2 x$

$v^2 = \omega^2(a^2 - x^2)$

$x = a\ \sin\omega t$

$T = \frac{2\pi}{\omega}$

ACTIVITIES:

1. A particle moves with S.H.M about a mean position O. When the particle is 30 cm from O, it is accelerating at 2 ms^{-2} towards O. Find the periodic time of the motion and the magnitude of the acceleration of the particle when 25 cm from O.
2. A particle moves with S.H.M about a mean position O. The particle has zero velocity at a point which is 60 cm from O and a speed of 4ms^{-1}. Determine:
   a) The maximum speed of the particle.
   b) The amplitude of the particle
   c) The periodic time of the particle.

LESSON 2: Relating elasticity and SHM

Introduction:

In the last lesson, we considered particles performing Simple Harmonic motion (S.H.M). In the topic of elasticity, we also covered forces acting when a spring is extended or compressed. We are now going to consider S.H.M in light springs.

Instruction:

When a spring on a horizontal smooth surface is fixed at one end and a mass attached to the other end, if the mass is pulled and released, it will oscillate and perform SHM.

By Hooke's law, we have, $T = \lambda\frac{x}{l}$ $\rightarrow$ *when we use* $F = ma$, *then* $- \lambda\frac{x}{l} = m\ddot{x}$

$$\therefore \ddot{x} = \frac{\lambda}{ml}x \quad \rightarrow \quad n = \sqrt{\frac{\lambda}{ml}}$$

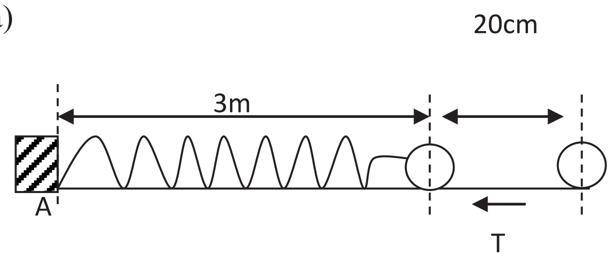
Example One:

One end of a light elastic spring of natural length 10m and modulus of elasticity 16N is fixed to a point A on a smooth horizontal surface. A body of mass 100g is attached to the other end of the spring and is held at rest at a point B on the surface, causing the spring to be extended by 20 cm.

(a) show that when released, the body will move with SHM.

(b) Find the period of the motion.

(a)



Suppose the spring is extended a distance x. A tension, T tends to pull it towards A.

By Hooke's law, we have, $T = \lambda\frac{x}{l} = 16 \times \frac{x}{10} = 1.6x$

Since T is opposite the increase in x,

resultant force on the body is $T = -ma \Rightarrow T = -\frac{100}{1000}a$ where a is the acceleration towards point A.

Therefore, $1.6x = -\frac{100}{1000}a$

And $a = -16x$ which is in the form $a = -n^2x$ which is SHM about the point where x is zero and n= 4.

(b) $n = \frac{2\pi}{T} \Rightarrow T = \frac{2\pi}{4}$

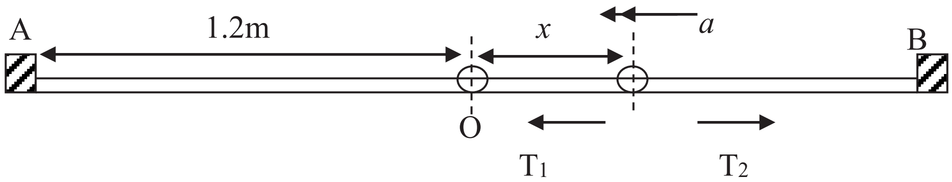
Therefore, $T = \frac{\pi}{2} rads^{-1}$

Example two.

A light elastic string of length 2m and modulus 18N is stretched between two points A and B 2.4 m apart on a smooth horizontal surface. A body of mass 2kg attached to the midpoint of the string is pulled 8cm towards B and then released.

(a) Show that the body moves in SHM.

(b) Find the speed of the body when it is 1.05m from B.



Let O be the centre of AB and the displacement from O be x. Each un-stretched length is 1m.

Force tending to reduce x is $T_1 - T_2 = \lambda\frac{(0.2+x)}{l} - \lambda\frac{(0.2-x)}{l}$

Therefore $\lambda\frac{(0.2+x)}{l} - \lambda\frac{(0.2-x)}{l} = -ma$

$$18\times\frac{(0.2+x)}{1} - 18\times\frac{(0.2-x)}{1} = -2\times a$$

$$18x = -2\times a \Rightarrow a = -9x$$

which is in the form $a = -n^2x$ which is SHM about the point where x is zero and n= 3.

(b). Using $v^2 = n^2(A^2 - x^2)$,amplitude, A = 0.08m and displacement $x = 0.05$m

$$v^2 = 9(0.08^2 - 0.05^2)$$
$$= 0.0351 \therefore v = 0.1873ms^{-1}$$

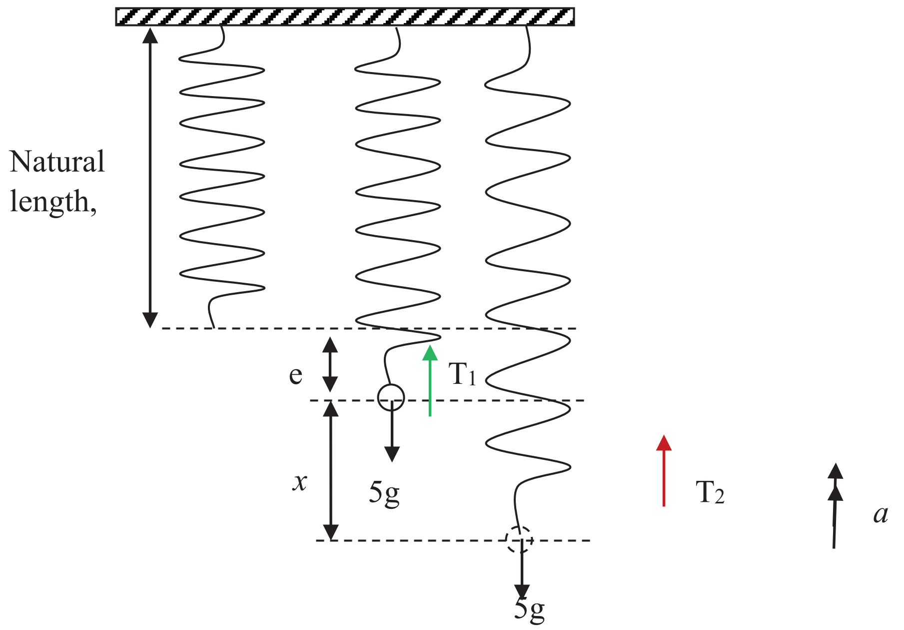
Example three

A light elastic spring of length 60cm and modulus 100N hangs vertically with its upper end fixed and a body of mas 5kg attached to its lower end . The body initially rests in equilibrium and is then pulled down a distance of 15cm and released.

(a) Show that the body executes SHM.

(b) Find the maximum velocity.

(a)



Consider the extension in equilibrium to be e.

Therefore the tension, $T_1 = \lambda \frac{e}{L} = 5g$, where L is the natural length.

When displaced, the resultant force that tends to decrease x is $T_2 - 5g = \lambda \frac{(x+e)}{L} - \lambda \frac{e}{L}$

Therefore,

$$100 \times \frac{(x+e)}{0.8} - 100 \times \frac{e}{0.8} = -5a$$

$$100 \times \frac{x}{0.8} = -5a \Rightarrow a = -25x$$

Which is in the form $a = -n^2 x$ which is SHM about the equilibrium point where x is zero and n=
5

(b) Maximum velocity is when $x = 0$.

Using $v^2 = n^2(A^2 - x^2)$,amplitude, A = 0.15m

$$v^2 = 25(0.15^2 - 0.00^2)$$

$$= 0.0351 \therefore v\max = 0.75ms^{-1}$$

ACTIVITIES:

1. A light spring is of natural length 2 m and modulus 4 N. One end of the spring is attached to a fixed point P on a smooth horizontal surface and to the other end is attached a body of mass 700g. The body is held at rest on the surface at a distance of 2.5 m from P. Show that on release the body will move with SHM and find the amplitude and periodic time of the motion.

2. A light spring of natural length 60 cm and modulus 150 N hangs vertically with its upper end fixed and a body of mass 3 Kg attached to the lower end. When the system is resting in equilibrium, the body is projected vertically downwards with a speed of 2.8 ms^{-1}. Show that the resulting motion will be simple harmonic and find the amplitude of the motion.

Term 3

Topic: CENTRE OF GRAVITY

In this topic you should be able to:

- apply moment of force in finding the centre of gravity (C.O.G) of a system of particles.
- find the centre of gravity of system of particles.
- find the C.O.G of uniform rods, disks, triangular and rectangular laminas, spheres, cuboids and combinations of bodies.
- calculate centre of gravity of the remaining portion of the body.

Materials

aruler**,** a manila/cardboard, a pen, foolscaps or note books,

LESSON 1 Finding Centre of Gravity of a system of particles:

Introduction:

In the topic of Moment of a force, we found moments about any given point. We also found the line of action of the resultant of given forces. We are now going to apply moment of force in finding the centre of gravity (C.O.G) of a system of particles.

Centre of Gravity of a system of particles:

The centre of gravity of any number of particles is the point through which the line of action of the resultant of their weights always passes.

We apply the principle of moments to find the C.O.G.

Principle of Moments:

When a number of coplanar forces act on a body, the algebraic sum of the moments of these forces about any point in their plane, is equal to the moment of the resultant of these forces about that point.

The position of the centre of gravity of a system of particles $m_1, m_2, m_3 \ldots m_n$ at the points $(x_1, y_1), (x_2, y_2), (x_3, y_3) \ldots (x_n, y_n)$ is at the point $(\bar{x}, \bar{y})$.

We apply the principle of moments about the *y- axis* to obtain the *x* coordinate:

$$m_1x_1 + m_2x_2 + m_3x_3 + \cdots + m_nx_n = (m_1 + m_2 + m_3 + \cdots + m_n)\bar{x}$$

$$\bar{x} = \frac{\sum(m_ix_i)}{\sum m_i}$$

Also we apply the principle of moments about the *x- axis* to obtain the *y* coordinate:

$$m_1x_1 + m_2x_2 + m_3x_3 + \cdots + m_nx_n = (m_1 + m_2 + m_3 + \cdots + m_n)\bar{y}$$

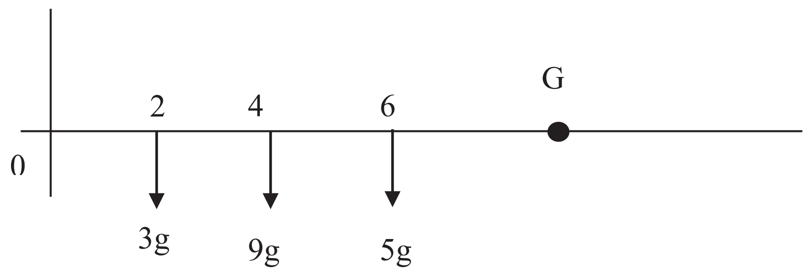
$$\bar{y} = \frac{\sum(m_iy_i)}{\sum m_i}$$

Using $v^2 = n^2(A^2 - x^2)$,amplitude, A = 0.08m and displacement $x = 0.05$m

$$v^2 = 9(0.08^2 - 0.05^2)$$
$$= 0.0351 \therefore v = 0.1873ms^{-1}$$

Example one

Three particles of mass 3kg, 9kg, and 5kg are located at points (2,0) , (4 , 0) and (6 , 0) respectively. Find their centre of gravity.



Let their centre of gravity be at G(X, Y)

Taking moments about O,

Sum of moments of the individual weights = moment of resultant weight.

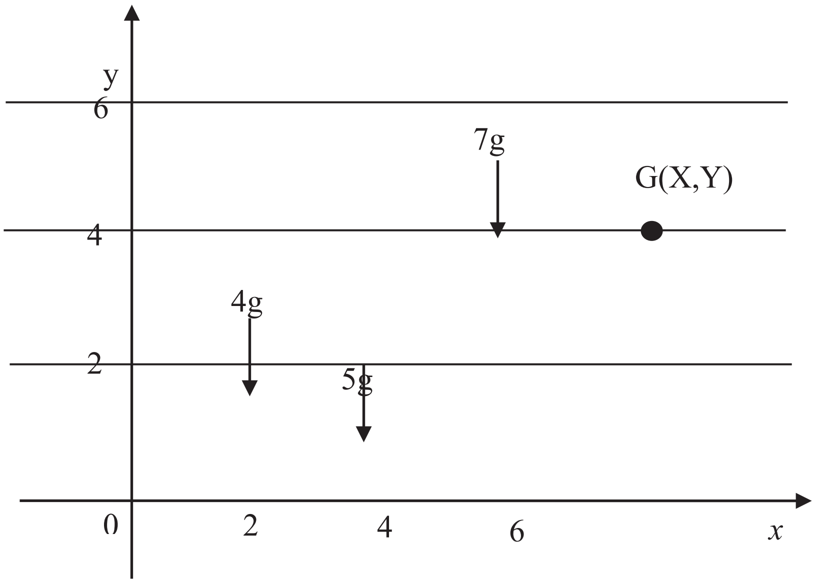
$$(3g \times 2) + (9g \times 4) + (5g \times 6) = (3g + 9g + 5g) \times X$$
$$\Leftrightarrow 72g = 17gX$$
$$\therefore X = \frac{72}{17}(4.2352)$$

For Y, they all lie on y=0 (a line of symmetry) So, G lies on the same line.

Therefore G(4.2352 , 0)

Example three.

Three particles of mass 4kg, 5kg, and 7kg are located at points (2,3) , (4 , 2) and (6 , 5) respectively. Find their centre of gravity.



Let their centre of gravity be at G(X, Y)

Taking moments about the y-axis,

Sum of moments of the individual weights = moment of resultant weight.

$$(4g \times 2) + (5g \times 4) + (7g \times 6) = (4g + 5g + 7g) \times X$$
$$\Leftrightarrow 70g = 16gX$$
$$\therefore X = \frac{70}{16}(or 4.375)$$

Taking moments about the y-axis,

$$(4g \times 3) + (5g \times 2) + (7g \times 5) = (4g + 5g + 7g) \times X$$
$$\Leftrightarrow 60g = 16gX$$
$$\therefore X = \frac{60}{16}(or 3.75)$$

Therefore G(4.375 , 3.75)

ACTIVITES:

1. Four particles of mass 6 kg, 3 kg, 3 kg, and 4 kg are situated at points (3,1), (4,3), (5,2) and (-3,1) respectively. Find the coordinates of the centre of gravity of the four particles.

2. Three particles of mass 4 kg, 2 kg and 6 kg are situated at the points (4,3), (1,0), and (p, q) respectively. If the centre of gravity of the system of the particles lies at (0, 2), determine the values of p and q.

3. Four particles of mass 4 kg, 2 kg, 6 kg and 4 kg are situated at the points with position vectors $(6\boldsymbol{i} + 6\boldsymbol{j})$, $(3i + 5\boldsymbol{j})$, $((7\boldsymbol{i} + 3\boldsymbol{j})$ and $(2\boldsymbol{i} - \boldsymbol{j})$ respectively. Determine the position of the centre of gravity of the particles.

LESSON 2 Finding C.O.G of rigid bodies

Introduction

In the last lesson, we applied moment of force in finding the centre of gravity (C.O.G) of a system of particles. We are now going to find the C.O.G of some rigid bodies and laminae.

C.O.G of rigid bodies

The position of the centre of gravity of some rigid bodies lies on the axis of symmetry of those bodies.

Uniform Bodies:

A **uniform body** is one in which equal volumes have equal masses or weight. The centre of gravity of a uniform body is at the centre of that body.

Lamina:

A lamina is a flat body whose thickness is negligible compared to its other dimensions. A **uniform lamina** is one in which equal areas have equal masses or weights.

Instruction
Get a ruler and balance it on your finger at different points.

Observation:
The ruler will balance in equilibrium in the middle which shows it is uniform. The centre of gravity of this ruler is in the middle and that is where its weight acts downward.

Examples of uniform bodies:

Uniform rod. C.G. lies at its centre

Uniform right circular cylinder. C.G lies along the axis midway between the ends.

Uniform triangular lamina. C.G lies on the median and two-thirds of the distance from the vertex to the other side of the triangle.

Uniform rectangular lamina. C.G. lies At the centre.

Uniform circular lamina. C.G. lies at the centre.

Uniform sphere C.G lies at its centre.

A uniform cone: C.G. lies at the a quarter way from the base.

ACTIVITIES:

Obtain some of the bodies or Laminae (shapes using manila) given above and try to find the point at which the body balances. Compare with the information given.

LESSON THREE: Finding C.O.G of Composite Lamina.

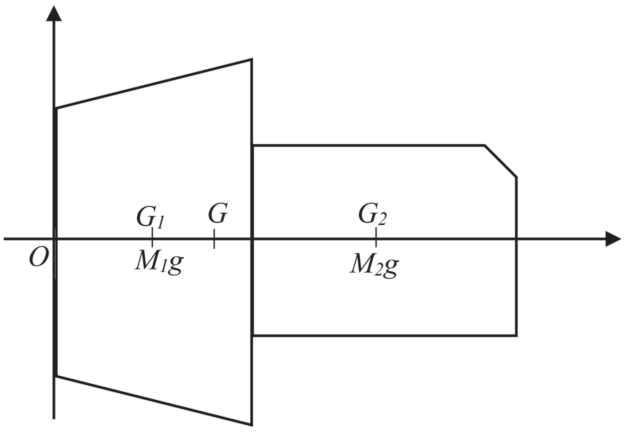
Introduction:

In the last lesson, we covered the C.O.G of some rigid bodies and lamnae. We are now going to consider the C.O.G of a combination of bodies or laminae (Plural for ‘lamina’). We shall refer to the combination of the laminae as a composite lamina.

C.O.G of Composite Laminae

A composite lamina consists of two or more regular laminae joined together. The centre of gravity of a composite lamina lies on the line of axis were the centre of gravities of each lamina lies.

We apply the principle of moments to find the centre of gravity of the composite body.

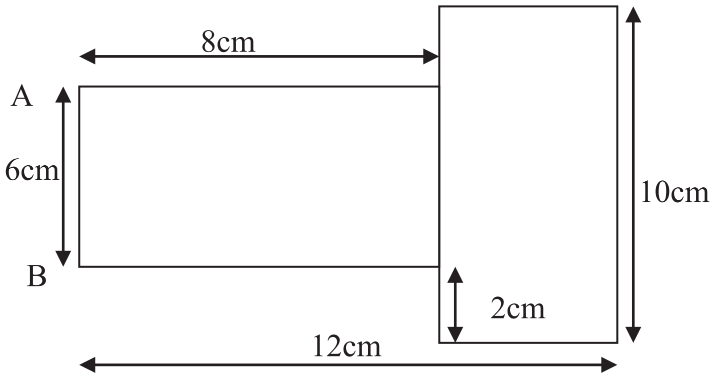


Lamina 1 has mass M_1 and C.O.G G_1
Lamina 2 has mass M_2 and C.O.G G_2
Composite laminae has mass $(M_1 + M_2)$ and C.O.G as G.

We take moments about O and we have:

$$(M_1 g \text{ x } OG_1) + (M_2 g \text{ x } OG_2) = (M_1 + M_2)\, g \text{ x } OG$$

Example one.



The figure shows a uniform composite lamina. Find the position of the centre of gravity.

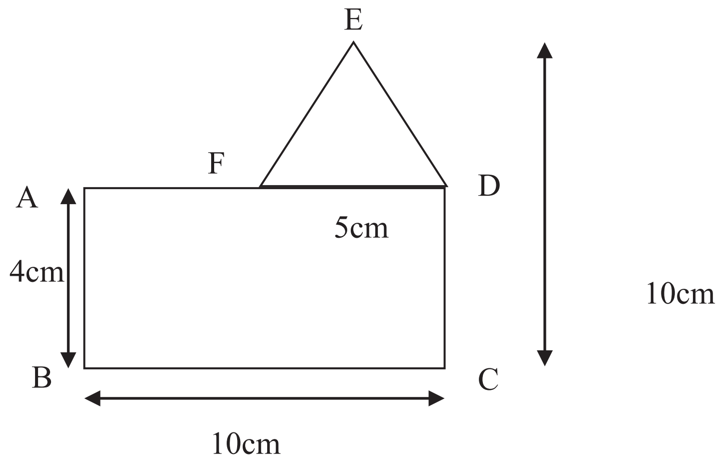
Let the mass per unit area be ρ.

Shape	Area	Mass	Moments about AB	Moments about centre.
Rectangle 1	8×6=48	48ρ	48ρ×4	0
Rectangle 2	4×10 = 40	40ρ	40ρ×10	0
Combined lamina		88ρ	88ρ×X	0

$592\rho = 88\rho X \quad \Rightarrow X = \dfrac{592}{88} = 6.7272cm$

Therefore CG is along the axis and 6.7272cm from AB.

Example two.



The figure shows a uniform composite lamina made of a rectangle ABCD and an isosceles triangle DEF. Find the position of the centre of gravity.

Let the mass per unit area be ρ.

Shape	Area(cm$^{2)}$	Mass	Moments about AB	Moments about BC.
Rectangle	4×10=40	40ρ	40ρ×5=200ρ	40ρ×2=80ρ
Triangle	$\frac{1}{2}\times 5\times 6=15$	15ρ	15ρ×7.5= 112.5ρ	15ρ×6= 90ρ
Total	55	55ρ	312.5ρ	170ρ
Combined lamina		55ρ	55ρ×X	55ρ×Y

$$55\rho X = 312.5\rho \Rightarrow X = \frac{312.5}{55} = 5.6818cm$$

$$55\rho Y = 170\rho \Rightarrow Y = \frac{170}{55} = 3.909cm$$

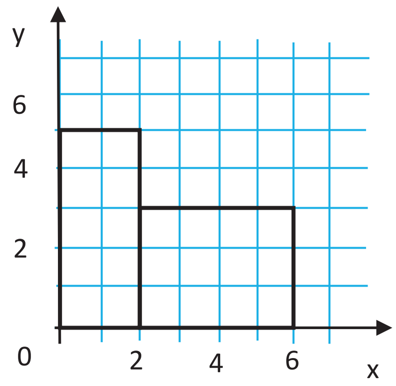
Therefore CG is 5.6818cm from AB and 3.909cm from BC..

(*Extracted from Understanding Mechanics*)

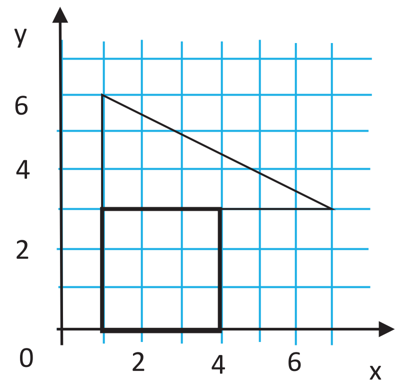
ACTIVITIES:

1. In each of the questions below, find the coordinates of the centre of gravity uniform composite laminae. Each grid consists of unit squares:

(a)



(b)



7. Two square laminae each of side 6 metres, are joined together to form a rectangular lamina, 12 metres by 6 metres. The squares are not made of the same material and the mass per unit area of one of the squares is three times the other. Find the distance of the centre of gravity of the composite laminae from the common edge of the squares.

LESSON FOUR: Finding C.O.G of remainder of a lamina

Introduction:

In the last lesson, we covered the C.O.G of a combination of bodies or laminae (Plural for 'lamina'). We are now going to calculate centre of gravity of the remaining portion of the body.

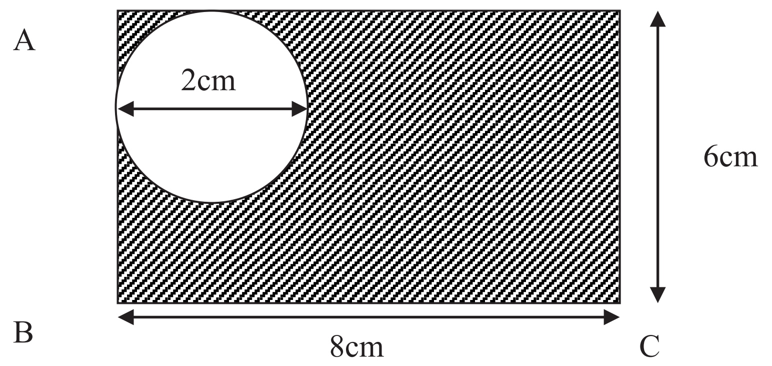
C.O.G of remainder of a lamina

A lamina can be perforated and a portion or portions of different shapes cut out. We then have to find the centre of gravity of the remaining portion of the laminae. As we did in the composite lamina, this time the known centre of gravity is for the complete lamina. The cut out portions and the remainder make up the complete laminae (composite).

$$(M_r g \text{ x } OG_r) = [(M_c + M_r)g \text{ x } OG] - (M_c g \text{ x } OG_c)$$

Where r-Remainder, c- Cut of portion and c+r- Complete laminae before perforation

Example:



The shaded area shows a uniform rectangular laminar with a circular hole. Find the centre of gravity of the laminar.

Let the mass per unit area be ρ.

Shape	Area(cm$^{2)}$	Mass	Moments about AB	Moments about BC.
Complete Rectangle	8×6=48	48ρ	48ρ×4= 192ρ	48ρ×3= 144ρ
Removed circle	$\pi\times1^2 = \pi$	πρ	πρ×1 =πρ	πρ× 5= 5 πρ
Remaining laminar	48 - π	(48 - π)ρ	(48 - π)ρX	(48 - π)ρ Y

Moment of remainder = moment of complete – moment of removed.

$192\rho - \pi\rho = (48 - \pi)\rho X$

$188.86 = 44.86X \qquad X = 4.21\text{cm}$

Also, $144\rho - 5\pi\rho = (48 - \pi)\rho Y$

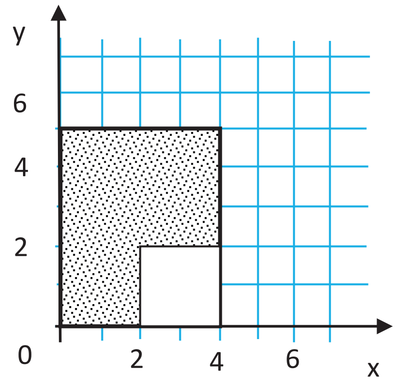
$128.29 = 44.86X \qquad Y = 2.86\text{cm}$

Therefore CG is 4.21cm from AB and 2.86 cm from BC..

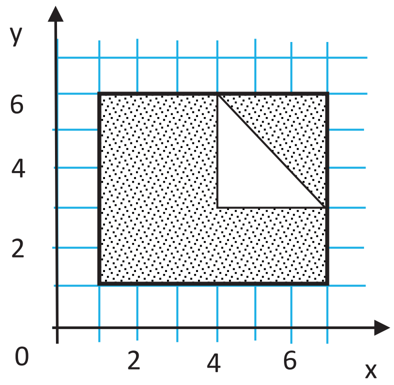
ACTIVITIES:

. Find the coordinates of the centre of gravity of each one of shaded remaining lamina shown. Each grid consists of unit squares.

(a)



(b)



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